Cutting Circles into Pseudo-segments and Improved Bounds for Incidences*

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Abstract

We show that n arbitrary circles in the plane can be cut into $O(n^{3/2+\varepsilon})$ arcs, for any $\varepsilon > 0$, such that any pair of arcs intersect at most once. This improves a recent result of Tamaki and Tokuyama [20]. We use this result to obtain improved upper bounds on the number of incidences between m points and n circles. An improved incidence bound is also obtained for graphs of polynomials of any constant maximum degree. BORIS SAYS: mention pseudosegments?

1 Introduction

Let P be a finite set of points in the plane and C a finite set of circles. Let $I = I(P, C)$ denote the number of incidences between the points and the circles. Let $I(m, n)$ denote the maximum value of $I(P, C)$, taken over all sets P of m points and sets C of n circles, and let $I'(m, n, X)$ denote the maximum value of $I(P, C)$, taken over all sets P of m points and sets C of n circles with at most X intersecting pairs.

In this paper we derive improved upper bounds for $I(m, n)$ and $I'(m, n, X)$. The previous best upper bounds were $I(m, n) = O(m^{3/5}n^{4/5}+m+n)$ [8, 15], and $I'(m, n, X) = O(m^{3/5}X^{2/5}+m+n)$ [4]. The bounds that we obtain are:

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & \text{for } m \geq n^{(5-3\varepsilon)/(4-9\varepsilon)} \\ O(m^{(6+3\varepsilon)/11}n^{(9-\varepsilon)/11} + n) & \text{for } m \leq n^{(5-3\varepsilon)/(4-9\varepsilon)} \end{cases}$$

and

$$I'(m,n,X) = \begin{cases} O(m^{2/3}X^{1/3} + m) & \text{for } m \geq X^{(1+6\varepsilon)/(4-9\varepsilon)}n^{3(1-5\varepsilon)/(4-9\varepsilon)} \\ O(m^{(6+3\varepsilon)/11}X^{(4+2\varepsilon)/11}n^{(1-5\varepsilon)/11} + n) & \text{for } m \leq X^{(1+6\varepsilon)/(4-9\varepsilon)}n^{3(1-5\varepsilon)/(4-9\varepsilon)}, \end{cases}$$

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for any $\varepsilon > 0$, where the constants of proportionality depend on ε. The bound on $I(m,n)$ is worst-case tight when m is larger than roughly $n^{5/4}$. This follows from the construction of $\Theta(m^{2/3}n^{2/3})$ incidences between m points and n lines (see, e.g., [10]) which, after applying an inversion to the plane, becomes a configuration with $\Theta(m^{2/3}n^{2/3})$ incidences between m points and n circles.

For sets C of circles with the additional property that every two circles in C intersect we estabilish the slightly improved bound

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & \text{for } m \geq n^{5/4} \\ O(m^{6/11}n^{9/11} + n) & \text{for } m \leq n^{5/4}. \end{cases}$$

Our results are strongly related to a theorem of Tamaki and Tokuyama [20], which asserts that n circles (or, more generally, n "pseudo-parabolas") can be cut into $O(n^{5/3})$ pseudo-segments, i.e., arcs with the property that each pair intersect at most once. In this paper we improve this result for the case of circles, showing that n circles can be cut into $O(n^{3/2+\varepsilon})$ arcs, for any $\varepsilon > 0$, so that each pair of arcs intersect at most once.

In the second part of the paper we combine the new bound with several other tools to derive the aforementioned improved bounds for $I(m,n)$ (and $I'(m,n,X)$). These tools are reviewed in Section 2.

Recently, Chan [5] has extended Tamaki and Tokuyama's result to the case of graphs of polynomials of any constant maximum degree. Using Chan's bound and extending, in a straightforward manner, the proof technique we used for the case of circles, we also obtain bounds for the number of incidences between m points and n such graphs. These bounds improve (slightly) those obtained earlier in [15].

2 Preliminaries

2.1 Lenses in a circle arrangement

Let C be a family of n circles of arbitrary radii in the plane. Let γ, γ' be two circles in C, which intersect at two points u, v. The union of an arc of γ and an arc of γ', each connecting u and v, is called a *lens*. Two lenses are said to *overlap* if an arc of one of them overlaps an arc of the other (necessarily lying on the same circle). The *interior* of the lens is the open bounded region enclosed by its two circle arcs. We say that another circle δ *crosses* the lens ℓ if it intersects both the lens itself (possibly at u and/or v) and its interior. The *level* of a lens is defined to be the number of circles of C that cross it. See Figure 1. *Empty* lenses are precisely the 0-level lenses. They are also faces of the arrangement $\mathcal{A}(C)$ of C that have degree 2. With the possible exception of the unbounded face, these faces are either *lens-faces* (contained in the interiors of the two defining circles) or *lune-faces* (contained in the interior of one defining circle and in the exterior of the other); see Figure 1. In recent papers, Pinchasi [16] and Alon et al. [4] have obtained various upper bounds for the number of these faces: It was shown that if every pair of circles in C intersect then the number of lens-faces and lune-faces is $O(n)$. In fact, the following stronger result, which is crucial for our analysis, was proven in [4]:

Lemma 2.1. *Let A and B be two families of circles in the plane, such that each circle in A intersects every circle in B, and there is a point p that is interior to all the circles of A. Then*

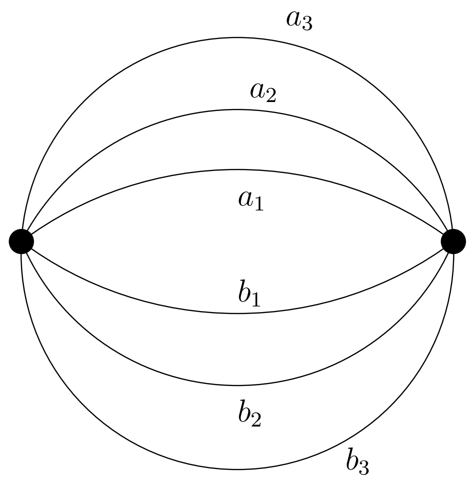

Figure 1: A pencil of circles that pass through two common points. There are five empty lenses, one of which is (a_1, b_1), which is a lens-face, whereas the other four are lune-faces. The lens (a_2, b_2) is at level 2 (it is crossed by circles containing a_1 and b_1).

the number of empty lenses within the family $A \cup B$ that are defined by a circle of A and by a circle of B is $O(|A| + |B|)$.

Alon et al. [4] have also shown that if all the circles of C have the same radius (but not all pairs necessarily intersect) then the number of lens-faces is $O(n^{4/3} \log n)$ and the number of lune-faces is at most n. If the circles have arbitrary radii, then the number of lens-faces and lune-faces is $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$. Clearly, this latter result is now subsumed by our new bound.

2.2 Crossing lemma

A simple graph is said to be *drawn* in the plane if its vertices are mapped to distinct points in the plane, and each of its edges is mapped to a curve connecting the points corresponding to the end vertices of the edge. We further require that no curve passes through any other vertex and that each pair of curves meet a finite number of times. A *crossing* between two curves is a point at which their relative interiors intersect transversally. An *edge-crossing* in (the drawing of) the graph is a pair of crossing edges.

Lemma 2.2 (Leighton [12]; Ajtai et al. [3]; see also [14]). *Any plane drawing of a simple graph G with e edges and n vertices must have $\Omega(e^3/n^2)$ edge-crossings, provided that $e \geq 4n$. Equivalently, if G can be drawn in the plane with X edge-crossings then $e = O(n^{2/3} X^{1/3} + n)$.*

Slightly abusing the notation, we will sometimes not distinguish between the vertices of a graph and the corresponding points in its plane drawing or between a graph edge and the curve that represents it in the drawing.

2.3 Simplicial partitioning of point sets in higher dimensions

We also need the following well-known result of Matoušek:

Theorem 2.3 (Matoušek [13]). *Let A be a set of n points in $\mathbb{R}^d$ and $1 \le r \le n$ a given parameter. Then A can be partitioned into $q \le 2r$ subsets, $A_1, \ldots, A_q$, so that, for each i, $|A_i| \le n/r$, and A_i is contained in a (possibly lower-dimensional) simplex Δ_i, so that no hyperplane* crosses *(i.e., intersects but does not contain) more than $O(r^{1-1/d})$ of these simplices.*

In addition, the following refinement of Theorem 2.3, due to Agarwal and Matoušek, will be useful in our analysis:

Theorem 2.4 (Agarwal–Matoušek [1]). *Let A be a set of n points in $\mathbb{R}^d$ that lie on an algebraic $(d-1)$-dimensional surface of fixed constant degree, and let $1 \le r \le n$ be a given parameter. Then A can be partitioned into $q \le 2r$ subsets, $A_1, \ldots, A_q$, so that, for each i, $|A_i| \le n/r$, and A_i is contained in a (possibly lower-dimensional) simplex Δ_i, so that no hyperplane* crosses *(i.e., intersects but does not contain) more than $O(r^{1-1/(d-1)} \log r)$ of these simplices.*

3 Cutting Circles into Pseudo-segments

In this section we are going to improve the following result.

Theorem 3.1 (Tamaki and Tokuyama [20, Theorem 6.1]). *Let C be a collection of n circles in the plane. Then the circles of C can be cut into a total of $O(n^{5/3})$ arcs, so that any two of these arcs intersect at most once.*

We note that this result is in fact given in [20] in more generality, and applies also to any collection of *pseudo-parabolas*, which are graphs of continuous totally-defined functions, any two of which intersect at most twice, and also to any collection of convex *pseudo-circles*, i.e., closed convex curves, any two of which intersect at most twice. At the moment we do not have any improved upper bound on the number of cuts in these more general settings, but we do have an improvement for the case of circles:

Theorem 3.2. *Let C be a collection of n circles of arbitrary radii in the plane. The circles of C can be cut into $O(n^{3/2+\varepsilon})$ arcs, for any $\varepsilon > 0$, so that any two of the arcs intersect at most once.*

A close inspection of the analysis of [20] reveals that it suffices to obtain the bound $O(n^{3/2+\varepsilon})$ for the maximum size $\nu_1(C)$ of a family $\mathcal{L}$ of pairwise nonoverlapping lenses in $\mathcal{A}(C)$. Once this bound is obtained, we can then plug it into the remainder of the analysis of [20] without any further modifications, and obtain the same asymptotic bound on the desired number of cuts. The assertion of the theorem is thus an immediate consequence of the following result:

Theorem 3.3. *The maximum size $\nu_1(C)$ of a family of pairwise nonoverlapping lenses in an arrangement of n circles of arbitrary radii is $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$.*

Proof. Let C be such a set of n circles and let $\mathcal{L}$ be a family of pairwise nonoverlapping lenses in $\mathcal{A}(C)$. We need to show that $|\mathcal{L}| = O(n^{3/2+\varepsilon})$.

Represent a circle c whose center lies at (a, b) and whose radius is r by the point

$$p_c(a, b, r, a^2 + b^2 - r^2) \in \mathbb{R}^4,$$

and by the pair of hyperplanes

$$h_c^+ : \ x_4 = 2ax_1 + 2bx_2 + 2rx_3 + (r^2 - a^2 - b^2)$$
$$h_c^- : \ x_4 = 2ax_1 + 2bx_2 - 2rx_3 + (r^2 - a^2 - b^2).$$

A circle c of radius r centered at (a, b) and a circle c' of radius R centered at (ξ, η) intersect if and only if

$$(R - r)^2 \leq (a - \xi)^2 + (b - \eta)^2 \leq (R + r)^2,$$

or

$$2a\xi + 2b\eta + 2rR + (r^2 - a^2 - b^2) \geq \xi^2 + \eta^2 - R^2$$

and

$$2a\xi + 2b\eta - 2rR + (r^2 - a^2 - b^2) \leq \xi^2 + \eta^2 - R^2.$$

In other words, they intersect if and only if the point $p_{c'}$ lies in the wedge above h_c^- and below h_c^+.

Partition C at random into two subfamilies A, B of equal size. The expected number of lenses in $\mathcal{L}$ that are 'bichromatic' (formed by a circle in A and a circle in B) is $|\mathcal{L}|/2$. It thus suffices to obtain an upper bound on the number of bichromatic lenses in $\mathcal{L}$, and in what follows we will assume that all lenses in $\mathcal{L}$ are bichromatic (the lenses in such a subset clearly continue to have the property that no two of them overlap).

Map the circles in A to their dual points in $\mathbb{R}^4$, and let A^* denote the resulting point set. Map each circle in B to the dual wedge in $\mathbb{R}^4$, and let B^* denote the resulting set of wedges.

Apply Theorem 2.4 to A^*, with a sufficiently large constant parameter r that will be specified below. We obtain a partitioning of A^* into $q \leq 2r$ subsets $A_1^*, \ldots, A_q^*$, each of size at most $n/(2r)$, so that any hyperplane crosses the convex hulls of at most $O(r^{2/3} \log r)$ subsets.

Fix a subset A_i^* and let B_i^* denote the set of wedges in B^* that fully contain A_i^*. Let A_i and B_i denote the corresponding sets of circles. By the properties of this transformation, each circle in A_i intersects every circle in B_i, but there may be disjoint pairs of circles in $A_i \times A_i$ and in $B_i \times B_i$. We next proceed as in [4]. That is, suppose, without loss of generality, that the smallest circle in $A_i \cup B_i$ is $c \in A_i$, and let r be the radius of c. Let D_0 be the disk of radius $3r$ concentric with c. Each circle $c' \in B_i$ intersects c and has radius $r' \geq r$, which is easily seen to imply that the intersection of D_0 with the disk D' that c' bounds has area at least πr^2. Hence, we can place $O(1)$ points in D_0 so that any such D' contains at least one of them. This implies that we can decompose B_i into $O(1)$ families $B_i^{(1)}, \ldots, B_i^{(p)}$ so that all the circles in the same family have a common point in their interiors.

Now fix a pair of families $A' = A_i$, $B' = B_i^{(j)}$, and consider the subtask of estimating the number of (bichromatic) lenses in $\mathcal{L}$ formed by a circle of A' and a circle of B'. First, we note that Lemma 2.1 implies that the number of bichromatic *empty* lenses (both those corresponding to lens-faces and to lune-faces) in $A' \cup B'$ is $O(|A'| + |B'|)$.

Next, we estimate the number of bichromatic lenses in $A' \cup B'$ whose level is at most k, for an appropriate threshold parameter k whose value will be fixed momentarily. See Figure 2 for an illustration. By a straightforward application of the Clarkson-Shor probabilistic analysis technique [7], the number L_k of bichromatic lenses at level at most k is $O(k^2)$ times the number of empty bichromatic lenses in a random sample of n/k circles of $A' \cup B'$. Using Lemma 2.1, $L_k = O(k^2 \cdot (N/k)) = O(Nk)$, where $N = |A'| + |B'|$.

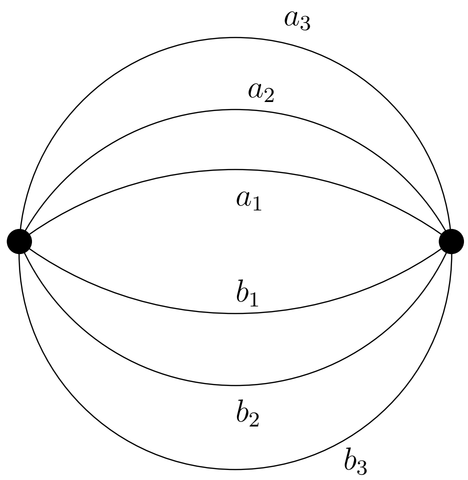

Figure 2: A pencil of circles of $A' \cup B'$ that pass through two common points. Circles a_1, a_2, a_3 belong to A' and circles b_1, b_2, b_3 belong to B'. The only empty bichromatic lens is (a_1, b_1). $\mathcal{L}$ may contain the lenses (a_1, b_1), (a_2, b_2), (a_3, b_3), at levels 0, 2, and 4, respectively. Or $\mathcal{L}$ may contain the lenses (a_1, b_3), (a_2, b_2), (a_3, b_1), all at level 2.

Let $\mathcal{L}^*$ be the subcollection of $\mathcal{L}$ consisting of all lenses in $\mathcal{A}(A' \cup B')$ with levels greater than k. Let $\ell \in \mathcal{L}^*$, and let $c \in A', c' \in B'$ be the two circles whose arcs form ℓ. Then ℓ can be naturally associated with at least k ordered pairs of crossing circles of the form (c, γ) or (c', γ), where γ is a circle in $A' \cup B'$ that crosses ℓ. We claim that any such pair (c, γ) can be associated with at most four lenses of $\mathcal{L}^*$. Indeed, let u, v denote the points of intersection of c and γ. Because of the pairwise nonoverlap condition, there can be at most two lenses in $\mathcal{L}^*$ that have an arc along c and contain u, and at most two lenses in $\mathcal{L}^*$ that have an arc along c and contain v. No other lens in $\mathcal{L}^*$ can be associated with (c, γ).

Hence the maximum number of pairwise nonoverlapping bichromatic lenses of level greater than k is $O(N^2/k)$. This, plus the bound $O(Nk)$ on all lenses of level at most k, yields the bound $O(Nk + N^2/k)$ on the number of lenses of $\mathcal{L}$ in $\mathcal{A}(A' \cup B')$. Substituting $k = N^{1/2}$, we obtain the bound $O(N^{3/2}) = O((|A_i| + |B_i^{(j)}|)^{3/2})$.

Summing this bound over the $O(1)$ indices j, we conclude that the number of bichromatic lenses in $A_i \cup B_i$ is $O((|A_i| + |B_i|)^{3/2})$. Summing this bound over all subsets A_i and corresponding subsets B_i, we conclude that the overall number of lenses in $\mathcal{L}$ of the type considered so far is $O(n^{3/2})$.

Any other bichromatic lens in $\mathcal{L}$ is formed by a circle c in some A_i and by a circle $c' \in B$ for which at least one of the half-hyperplanes bounding its dual wedge crosses the convex hull of A_i^*. Let us denote by $\bar{B}_i$ the subset of these circles, and put $m_i = |\bar{B}_i|$. By the properties of our decomposition, we have $\sum_{i=1}^{q} m_i = O(nr^{2/3} \log r)$. As a matter of fact, we can split each $\bar{B}_i$ into subsets of size at most $(n \log r)/r^{1/3}$ and duplicate the corresponding sets A_i. The number of new pairs $(A_i, \bar{B}_i)$ is still $O(r)$.

We next apply a symmetric decomposition step, using the same parameter r, to each pair $(A_i, \bar{B}_i)$, where now the circles of $\bar{B}_i$ are mapped into points in $\mathbb{R}^4$ and the circles of A_i are mapped into wedges. Repeating the entire process for each $(A_i, \bar{B}_i)$, and recalling that r is a constant, we obtain the following recurrence:

$$F(m) = O(m^{3/2}) + \beta r^2 F\left(\frac{m \log r}{r^{4/3}}\right), \tag{1}$$

for some absolute constant $\beta > 0$, where $F(m)$ is the maximum number of bichromatic lenses in $\mathcal{L}$ that can be formed between two subfamilies of circles of A and of B, respectively, whose cardinalities are both at most m; the constant of proportionality of the overhead term $O(m^{3/2})$ of (1) depends on r. It is now easy to verify that the solution to (1) is $F(m) = O(m^{3/2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε. This completes the proof of Theorems 3.3 and 3.2. $\square$

Theorem 3.2 can be strengthened as follows:

Theorem 3.4. *Let C be a set of n circles with at most X intersecting pairs. Then the circles of C can be cut into $O(n^{1/2-\varepsilon}X^{1/2+\varepsilon}+n)$ arcs, for any $\varepsilon > 0$, so that any two arcs intersect at most once.*

Proof. We assume that $X \geq n$. Otherwise, cutting each circle between each consecutive pair of points of its intersection with the other circles yields a collection of $O(n)$ arcs with the desired property. Put $r = \lceil n^2/X \rceil$, and let R be a random sample of r circles from C. Let $\mathcal{A}^*(R)$ denote the vertical decomposition of the arrangement $\mathcal{A}(R)$. It is obtained from $\mathcal{A}(R)$ by drawing a vertical segment through every vertex of $\mathcal{A}(R)$ and through every leftmost and rightmost point of a circle of R and extending it upward and downward until the first intersection with an edge of $\mathcal{A}(R)$ or to infinity, otherwise. This is a decomposition of the plane into "pseudo-trapezoidal" cells, whose expected number is $O(r + (r/n)^2 X) = O(r)$. For each cell τ, let C_τ denote the subset of circles of C that cross τ and put $n_\tau = |C_\tau|$. We first cut each circle of $C \setminus R$ at points in the interior of τ, slightly after it enters and alightly before it leaves cells of $\mathcal{A}^*(R)$, and then, for each cell τ, we cut further the portions of the circles of C_τ that lie inside τ into $O(n_\tau^{3/2+\varepsilon})$ subarcs, for any $\varepsilon > 0$, as in Theorem 3.2. We also cut the circles of R slightly before and after vertices of $\mathcal{A}^*(R)$. It is easily verified that this process does indeed cuts the circles into arcs, no pair of which intersect twice. The total number of arcs is thus $O(r) + \sum_\tau O(n_\tau^{3/2+\varepsilon})$. Using the results of [7], the expected value of this sum is

$$O(r) + O(r) \cdot O((n/r)^{3/2+\varepsilon}) = O(n^{3/2+\varepsilon}/r^{1/2+\varepsilon}) = O(n^{1/2-\varepsilon}X^{1/2+\varepsilon}),$$

for any $\varepsilon > 0$, as asserted. $\square$

Remark 3.5. *While Theorems 3.2 and 3.4 constitute significant improvements over the result of Tamaki and Tokuyama [20], they still leave a gap between the bounds that they establish and the best known lower bound $\Omega(n^{4/3})$, noted in [20]. We conjecture that the true bound is close to this lower bound.*

We close this section by observing that if C is a collection of n circles, every pair of which intersect, then we can slightly improve the preceding results. Specifically, we have:

Theorem 3.6. *If C is a collection of n circles, every two of which intersect, then one can cut the circles of C into $O(n^{3/2})$ subarcs, every two of which intersect at most once.*

Proof. The proof of Theorem 3.3 shows that $\nu_1(C) = O(n^{3/2})$. Indeed, there is no need to apply the recursive partitioning based on Theorem 2.4, because every pair of circles already intersect. $\square$

3.1 The case of polynomial curves

Let Γ be a collection of n curves that are graphs of polynomial functions of constant maximum degree s. In a recent paper, Chan [5] studied the problem of cutting the curves of Γ into arcs, each pair of which intersect at most once. The following bound, which is lower by a polylogarithmic factor than the bound given there, can be derived in a straightforward manner by the technique of [5]; it is smaller because we do not require the additional property that the resulting arcs be *extendible* to a collection of pseudolines, a property needed in Chan's application.

Theorem 3.7 (Chan [5]). *Any collection of n curves that are graphs of polynomial functions of constant maximum degree s can be cut into $O(n^{2-1/3^{s-1}})$ arcs, every two of which intersect at most once.*

Remark 3.8. *Chan's proof reduces the case at hand to that of cutting* parabolas *(given by equations of the form $y = ax^2+bx+c$) into pseudo-segments. Improving Tamaki and Tokuyama's bound for real parabolas will yield a parallel improvement of the bound of Theorem 3.7.*

4 Improved Bounds for Incidences Between Points and Circles

4.1 Improved bounds for many points

Let P be a set of m points in the plane and C a set of n circles with X intersecting pairs. Put $I = I(P, C)$.

By Theorem 3.4, we can cut the circles of C into $O(n^{1/2-\varepsilon}X^{1/2+\varepsilon})$ arcs, for any $\varepsilon > 0$, so that each two arcs intersect at most once. Let C' denote the resulting collection of arcs.

We draw a graph G in the plane whose vertices are the points of P, and whose edges connect pairs of points u, v that are consecutive along an arc of C'. We assume that each arc of C' contains at least two points of P; the contribution of the remaining arcs to I is at most $|C'|$. We also assume that any circle that has not been cut at all contains at least three points of P; the contribution of the other circles to I is at most $2n$. It is easily seen that the number e of edges of G is at least $I - cn^{1/2-\varepsilon}X^{1/2+\varepsilon} - 2n$, for some constant c. By construction, the graph G is simple, so the Crossing Lemma 2.2 implies that $I - cn^{1/2-\varepsilon}X^{1/2+\varepsilon} - 2n = O(m^{2/3}X^{1/3} + m)$. That is, $I = O(m^{2/3}X^{1/3} + n^{1/2-\varepsilon}X^{1/2+\varepsilon} + m + n)$. In other words, we have shown:

Theorem 4.1. *The maximum number of incidences between m points and n circles in the plane, with X crossing pairs of circles, is*

$$I'(m, n, X) = O(m^{2/3}X^{1/3} + n^{1/2-\varepsilon}X^{1/2+\varepsilon} + m + n), \tag{2}$$

for any $\varepsilon > 0$. In particular, the maximum number of incidences between m points and n circles in the plane is

$$I(m, n) = O(m^{2/3}n^{2/3} + n^{3/2+\varepsilon} + m), \tag{3}$$

for any $\varepsilon > 0$.

Remark 4.2. *As already argued in the introduction, the bound in (3) is tight when $m \geq n^{5/4+3\varepsilon/2}$.*

4.2 Improved bounds for any number of points

The bounds obtained above are $O(m^{2/3}n^{2/3} + m)$ when m is larger than roughly $n^{5/4}$ and $O(m^{2/3}X^{1/3}+m)$ when m is larger that roughly $X^{1/4}n^{3/4}$, but the bounds are larger for smaller values of m. Our next step is to use a partitioning of dual space to improve the bound for smaller values of m.

We use the following transformation, different from the one used in Section 3: A circle γ in the plane, of radius ρ and centered at (a,b), is mapped to the point $\gamma^*(a,b,a^2+b^2-\rho^2) \in \mathbb{R}^3$, and a point $p(\xi,\eta)$ in the plane is mapped to the plane $p^*: \ z = 2\xi x + 2\eta y - (\xi^2+\eta^2)$ in $\mathbb{R}^3$. As is easily verified, a point p lies on a circle γ if and only if the dual plane p^* contains the dual point γ^*. Let P^* denote the set of planes dual to the points of P and let C^* denote the set of points dual to the circles of C. No three planes of P^* pass through a common line, as all planes of P^* are tangent to the paraboloid $z = x^2+y^2$.

Apply Theorem 2.3 to C^* (with $d=3$), with a value of r that will be fixed shortly, to obtain a partitioning of C^* into subsets $C_1^*,\ldots,C_q^*$, where $q \le 2r$, with the properties stated in that theorem. Let C_i be the subset of circles in C that are dual to the points of C_i^*, let P_i denote the set of points of P whose dual planes cross the corresponding simplex Δ_i, and put $m_i = |P_i|$, for $i = 1,\ldots,q$. We have $\sum_{i=1}^q m_i = O(mr^{2/3})$.

Let $p \in P$ be a point that is incident to at least one circle in C_i, for some i. Then either the dual plane p^* crosses Δ_i, that is, $p \in P_i$, or p^* contains Δ_i. The latter case can arise only when Δ_i (and C_i^*) is not full-dimensional. If Δ_i has dimension 2 then there can be at most one point $p \in P$ whose dual plane contains Δ_i. If Δ_i is one-dimensional then, as noted above, there can be at most 2 points in P whose dual planes contain Δ_i. We may rule out the case that Δ_i is zero-dimensional, because then C_i is a singleton, and the construction of [13] never produces singleton subsets. Hence, the total number of incidences between P and C that fall into these degenerate categories is at most $2n$. In other words, we have shown that

$$I(P,C) \le 2n + \sum_{i=1}^q I(P_i,C_i).$$

Applying Theorem 4.1 to each $I(P_i,C_i)$, we thus obtain

$$I(P,C) = O\left(n + \sum_{i=1}^q \left(m_i^{2/3}(n/r)^{2/3} + (n/r)^{3/2+\varepsilon} + m_i\right)\right),$$

for any $\varepsilon > 0$, which, using Hölder's inequality, becomes

$$I(P,C) = O\left(n + \left(\sum_{i=1}^q m_i\right)^{2/3} \cdot r^{1/3} \cdot (n/r)^{2/3} + n^{3/2+\varepsilon}/r^{1/2+\varepsilon} + \sum_{i=1}^q m_i\right) =$$

$$O\left(n + \left(mr^{2/3}\right)^{2/3} \cdot r^{1/3} \cdot (n/r)^{2/3} + n^{3/2+\varepsilon}/r^{1/2+\varepsilon} + mr^{2/3}\right) =$$

$$O\left(m^{2/3}n^{2/3}r^{1/9} + n^{3/2+\varepsilon}/r^{1/2+\varepsilon} + mr^{2/3} + n\right).$$

Choose

$$r = \frac{n^{(15+18\varepsilon)/(11+18\varepsilon)}}{m^{12/(11+18\varepsilon)}}.$$

We note that $r \geq 1$ when $m \leq n^{5/4+3\varepsilon/2}$, which is the range under consideration (it is the range where $m^{2/3}n^{2/3} \leq n^{3/2+\varepsilon}$), and that $r \leq n$ provided that $m \geq n^{1/3}$, which we may also assume, since for smaller values of m we have $I(P,C) = O(n)$, as follows, e.g., from [8]. Substituting the value of r yields $I(P,C) = O(m^{(6+12\varepsilon)/(11+18\varepsilon)}n^{(9+14\varepsilon)/(11+18\varepsilon)} + n)$. This can be rewritten as $O(m^{(6+3\delta)/11}n^{(9-\delta)/11} + n)$, for $\delta = 8\varepsilon/(11+18\varepsilon)$. Replacing δ by ε, we thus obtain, as is easily checked, the following main result:

Theorem 4.3. *The maximum number of incidences between m points and n circles in the plane is*

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & \text{for } m \geq n^{(5-3\varepsilon)/(4-9\varepsilon)} \\ O(m^{(6+3\varepsilon)/11}n^{(9-\varepsilon)/11} + n) & \text{for } m \leq n^{(5-3\varepsilon)/(4-9\varepsilon)}, \end{cases} \tag{4}$$

for any $\varepsilon > 0$, where the constants of proportionality depend on ε.

Remark 4.4. *Our analysis only requires that the bound $\sum_i m_i = O(mr^{2/3})$ holds, rather than the stronger property, provided in Theorem 2.3, that each m_i is $O(r^{2/3})$. This weaker property can be derived in a manner that is somewhat simpler than the analysis of [13].*

We next extend the above analysis to derive an improved bound on $I'(m,n,X)$. We argue as in the proof of Lemma 3.4. That is, we fix $r = \lceil n^2/X \rceil$. We may assume, as above, that $X = \Omega(n)$, for otherwise, trivially, $I(m,n) = O(m+X) = O(m+n)$. Let R be a random sample of r circles from C. Let $\mathcal{A}^*(R)$ denote the vertical decomposition of the arrangement $\mathcal{A}(R)$, whose expected size is, as above, $O(r)$. For each (open) cell τ, let C_τ denote the subset of circles of C that either cross τ or contribute an arc to $\partial\tau$, and let P_τ denote the set of points of P in the closure of τ. Put $m_\tau = |P_\tau|$ and $n_\tau = |C_\tau|$. By construction, $I(P,C) \leq \sum_\tau I(P_\tau, C_\tau)$. Since a point can belong to at most two cells of which it is not a vertex, it follows that the expected value of $\sum_\tau m_\tau$ is at most $2m + O(r) = O(m)$, provided that $r \leq m$. Hence $I(P,C)$ is at most proportional to the expected value of the following expression (where, to simplify the notation, we have written the value of the threshold exponent $(5-3\varepsilon)/(4-9\varepsilon)$ as $5/4+\delta$):

$$\begin{aligned}
\sum_{m_\tau \geq n_\tau^{5/4+\delta}} & \left(m_\tau^{2/3} n_\tau^{2/3} + m_\tau\right) + \sum_{m_\tau < n_\tau^{5/4+\delta}} \left(m_\tau^{(6+3\varepsilon)/11} n_\tau^{(9-\varepsilon)/11} + n_\tau\right) \\
&= O(m) + \sum_\tau n_\tau + \sum_{m_\tau \geq n_\tau^{5/4+\delta}} m_\tau^{2/3} n_\tau^{2/3} + \sum_{m_\tau < n_\tau^{5/4+\delta}} m_\tau^{(6+3\varepsilon)/11} n_\tau^{(9-\varepsilon)/11} \\
&\leq O(m) + \sum_\tau n_\tau + \sum_\tau m_\tau^{2/3} n_\tau^{2/3} + \sum_\tau m_\tau^{(6+3\varepsilon)/11} n_\tau^{(9-\varepsilon)/11} \\
&\leq O(m) + \sum_\tau n_\tau + \left(\sum_\tau m_\tau\right)^{2/3} \left(\sum_\tau n_\tau^2\right)^{1/3} \\
&\quad + \left(\sum_\tau m_\tau\right)^{(6+3\varepsilon)/11} \left(\sum_\tau n_\tau^{(9-\varepsilon)/(5-3\varepsilon)}\right)^{(5-3\varepsilon)/11} \\
&\leq O(m) + O\left(r \cdot \frac{n}{r}\right) + O(m^{2/3}) \cdot O\left(r \cdot \left(\frac{n}{r}\right)^2\right)^{1/3} \\
&\quad + O(m^{(6+3\varepsilon)/11}) \cdot O\left(r \cdot \left(\frac{n}{r}\right)^{(9-\varepsilon)/(5-3\varepsilon)}\right)^{(5-3\varepsilon)/11} \\
&= O\left(\frac{m^{2/3} n^{2/3}}{r^{1/3}} + \frac{m^{(6+3\varepsilon)/11} n^{(9-\varepsilon)/11}}{r^{(4+2\varepsilon)/11}} + m + n\right),
\end{aligned}$$

where we have used Hölder's inequality and Clarkson and Shor's estimate [7] on the expected value of sums of the form $\sum_\tau n_\tau^\alpha$. Substituting the value of r, and assuming it to be at most m, we conclude that

$$I'(m,n,X) = O\left(m^{2/3} X^{1/3} + m^{(6+3\varepsilon)/11} X^{(4+2\varepsilon)/11} n^{(1-5\varepsilon)/11} + m + n\right).$$

If $r > m$ then the expected value of $\sum_\tau m_\tau$ is $O(r)$. If we substitute this bound in the above chain of inequalities, we obtain that

$$I(P,C) = O(r + n + r^{1/3} n^{2/3} + r^{(2+\varepsilon)/11} n^{(9-\varepsilon)/11}) = O(n),$$

since $r \leq n$. Hence the above bound for $I'(m,n,X)$ applies in all cases.

We can summarize the preceding arguments as follows.

Theorem 4.5.

$$I'(m,n,X) = \begin{cases} O(m^{2/3} X^{1/3} + m) & \text{for } m \geq X^{(1+6\varepsilon)/(4-9\varepsilon)} n^{3(1-5\varepsilon)/(4-9\varepsilon)} \\ O(m^{(6+3\varepsilon)/11} X^{(4+2\varepsilon)/11} n^{(1-5\varepsilon)/11} + n) & \text{for } m < X^{(1+6\varepsilon)/(4-9\varepsilon)} n^{3(1-5\varepsilon)/(4-9\varepsilon)}, \end{cases}$$

for any $\varepsilon > 0$, assuming that $X \geq n$; otherwise $I'(m,n,X) = O(m+X) = O(m+n)$.

4.3 Improved bound for the case of pairwise intersecting circles

Suppose that C is a collection of n circles, every pair of which intersect. In this case we can obtain a stronger bound on $I(P,C)$, by applying the analysis presented in the preceding subsections, replacing the bound $O(n^{3/2+\varepsilon})$ on the number of cuts by the slightly smaller bound $O(n^{3/2})$ provided in Theorem 3.6. Proceeding as above, we obtain the following slight improvement, as is easily verified.

Theorem 4.6. *The maximum number of incidences between m points and n pairwise-intersecting circles in the plane is*

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & m \geq n^{5/4} \\ O(m^{6/11}n^{9/11} + n) & m \leq n^{5/4}. \end{cases} \tag{5}$$

4.4 Incidences between points and graphs of polynomials

Let P be a set of m points and Γ be a collection of n curves that are the graphs of polynomials of degree at most s, for some fixed parameter $s \geq 1$. We wish to bound the number of incidences $I(P, \Gamma)$ between the points of P and the curves in Γ. We set $I(m,n) = \max\ I(P,\Gamma)$, where the maximum is taken over all sets P, Γ as above.

Our first step is similar to the analysis in Theorem 4.1. That is, we apply Chan's result, given in Theorem 3.7, to obtain a cutting of the curves in Γ into $O(n^{2-1/3^{s-1}})$ arcs, each pair of which intersect at most once. Continuing as in the proof of Theorem 4.1, we readily obtain the first bound

$$I(m,n) = O(m^{2/3}n^{2/3} + m + n^{2-1/3^{s-1}}). \tag{6}$$

As above, this bound is tight for $m \geq n^{2-1/(2\cdot 3^{s-2})}$.

To obtain an improved bound for smaller values of m, we apply the following duality transform. Each curve γ of the form $y = a_0 + a_1x + a_2x^2 + \cdots + a_sx^s$ is mapped to the point $\gamma^*(a_0, a_1, \ldots, a_s) \in \mathbb{R}^{s+1}$. Each point $p(\xi, \eta)$ is mapped to the hyperplane $p^* :\ x_0 + \xi x_1 + \xi^2 x_2 + \cdots + \xi^s x_s = \eta$. Clearly, incidences between points and curves are mapped to incidences between the corresponding dual hyperplanes and points.

Now apply Theorem 2.3 to the set Γ^* of points dual to the curves in Γ, with a parameter r that will be chosen shortly. We obtain a partition of Γ^* into $q \leq 2r$ subsets, $\Gamma_1^*, \ldots, \Gamma_q^*$, each containing at most n/r points, so that no hyperplane can be incident to points in more than $O(r^{s/(s+1)})$ subsets, except for subsets that the hyperplane fully contains. It is easily checked that if a subset has at least two points (a condition that always holds in the construction provided in Theorem 2.3) then the number of hyperplanes of the above form that fully contain the set is at most s. (In the primal plane, this statement asserts that two distinct polynomials of degree at most s cannot coincide at more than s points.) It follows that the number of incidences of the latter kind is at most $sn = O(n)$, so we may disregard them in what follows.

Applying the bound of (6) to each of the subsets Γ_i that correspond to the dual sets Γ_i^*, and summing over all such subsets, we obtain the bound

$$I(m,n) = O\left(n + \sum_{i=1}^{q}\left(m_i^{2/3}(n/r)^{2/3} + m_i + (n/r)^{\beta_s}\right)\right),$$

where $\beta_s = 2 - 1/3^{s-1}$ and where m_i is the number of hyperplanes dual to the points of P that cross the simplex containing Γ_i. Using Hölder's inequality and the fact that $\sum_i m_i = O(mr^{s/(s+1)})$, we have

$$\begin{aligned} I(m,n) &= O\left((mr^{s/(s+1)})^{2/3}r^{1/3}(n/r)^{2/3} + mr^{s/(s+1)} + \frac{n^{\beta_s}}{r^{\beta_s - 1}}\right) \\ &= O\left(m^{2/3}n^{2/3}r^{(s-1)/(3(s+1))} + mr^{s/(s+1)} + \frac{n^{\beta_s}}{r^{\beta_s-1}}\right). \end{aligned}$$

We now put

$$r = \frac{n^{\frac{4-\frac{1}{3^s-2}}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}}}{m^{\frac{2}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}}}.$$

We may assume that $n^{1/(s+1)} \le m \le n^{2-1/(2\cdot 3^{s-2})}$. Indeed, for $m > n^{2-1/(2\cdot 3^{s-2})}$ we already have a tight bound, and if $m < n^{1/(s+1)}$ then the number of polynomials that pass through at least $s+1$ of the given points is at most $O(m^{s+1}) = O(n)$, which is also easily seen to bound the number of incidences between these polynomials and the given points, whereas any other polynomial has at most s incidences with the given points, for an overall bound of $O(n)$ incidences. It is easily verified that in the assumed range for m we have $1 \le r \le n$. Substituting this value of r, the above bound becomes

$$I(m,n) = O\left(m^{\frac{2-\frac{2}{3^s-1}}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}} n^{1-\frac{\frac{2}{s+1}\left(1-\frac{1}{3^s-1}\right)}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}} + n \right).$$

One can also verify that this bound is (slightly) better than the bound

$$I(m,n) = O(m^{(s+1)/(2s+1)} n^{2s/(2s+1)} + m + n)$$

obtained in [15] (for somewhat more general families of curves), provided that $m > n^{1/(s+1)}$, which, as above, can be assumed.

We summarize this subsection in the following theorem.

Theorem 4.7. *The maximum number of incidences between m points and n graphs of polynomials of constant maximum degree s is*

$$I(m,n) = \begin{cases} O\left(m^{\frac{2-\frac{2}{3^s-1}}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}} n^{1-\frac{\frac{2}{s+1}\left(1-\frac{1}{3^s-1}\right)}{4-\frac{1}{3^s-2}-\frac{2}{s+1}}} + n \right) & m \le n^{2-1/(2\cdot 3^{s-2})} \\ O(m^{2/3}n^{2/3} + m) & m \ge n^{2-1/(2\cdot 3^{s-2})}. \end{cases} \qquad (7)$$

5 Conclusion

The main observation in this paper is that two recent analysis techniques, of Székely [19] and of Tamaki and Tokuyama [20], can be combined in a straightforward manner to yield improved incidence bounds for points and circles and for other families of curves. We suspect (and hope) that similar ideas can be applied to other related problems, in two or in higher dimensions.

This paper raises many open problems. We mention here some of the more obvious ones:

- Can the Tamaki-Tokuyama bound be further improved for circles? Can be improved at all for general pseudo-parabolas? for real parabolas? See the discussion following the statement of Theorem 3.1 and Remark 3.5.

- Can the technique of this paper be adapted to tackle the problem of the number of *distinct distances* in a set of n points in the plane? The setup in this problem involves a collection of many circles that have relatively few centers (see [19] and [18] for details).

- Find applications of the new incidence bounds obtained in this paper. Two problems to which the new bounds might be applicable are the unit distance problem in three dimensions [8] and the problem of bounding the maximum number of simplices spanned by a set of n points in $\mathbb{R}^d$ and congruent to a given simplex (see [2] for work in progress on this problem).

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