Supplementary Material

A. Proof of Lemma 5.4

The proof builds upon Lemmas A.1 and A.2 given below.

Lemma A.1. *In a residual flowgraph of a (a, b)-gadget based max flow problem with pixels p and q belonging to a clique* $\mathbf{c}$*:*

- *If there is an unsaturated path fragment from q_a to p_a in a gadget, then there must exist an unsaturated path fragment from p_b to q_b within the same gadget.*
- *If there exists an unsaturated path fragment from q_a to p_b in a gadget, then there must exist an unsaturated path fragment from p_a to q_b in the same gadget.*

Proof. Note that in any labeling configuration $\mathbf{l_c}$ corresponding to a clique $\mathbf{c}$, a pixel must be either labeled a or b (a pixel can not be labeled *both* a and b, or *neither* a nor b). Therefore, exactly one of the conjugate edges corresponding to p_a or p_b must participate in each DFC of the gadget. We prove the lemma for the first case. Similar arguments can be applied for the second case.

Since p and q belong to a clique, there exists at least one (a, b)-gadget containing nodes p_a, q_a, p_b and q_b. The existence of unsaturated path fragment from q_a to p_a implies:

1. There does not exist any tight DFC covering p_a and q_b. Since there is an unsaturated path fragment from q_a to p_a, the conjugate edge corresponding to p_a in the (a, b)-gadget is either not saturated, or if it is, then the DFC covering it also covers edge corresponding to q_a. Since all DFCs cover either q_a or q_b, it follows that there does not exist any tight DFC covering p_a and q_b. This is because any such tight DFC would not cover q_a and would force the path fragment from q_a to p_a to be saturated.
2. Conjugate edge incident at q_b is unsaturated, or if not then the tight DFC covering q_b covers p_b as well. If the edge incident at q_b is saturated, then all tight DFCs covering q_b must cover p_b as well. Since we have already shown that there does not exist any tight DFC covering q_b and p_a, all DFCs must cover either p_a or p_b.

Since conjugate edge incident at q_b is unsaturated or all tight DFCs cover both p_b and q_b, the path fragment from p_b to q_b is therefore unsaturated. □

Lemma A.2. *In a residual flowgraph of a (a, b)-gadget based max flow problem and a pixel p:*

- *If there exists a path of length l from s to p_a, then there must exist a path of length l from p_b to t.*
- *If there exists a path of length l from s to p_b, then there must exist a path of length l from p_a to t.*

Proof. For a pixel p let the length of the path in terms of the number of path fragments from s to node p_a be denoted by $l(s, p_a)$, and the length of the path from t to p_b by $l(t, p_b)$. The lemma holds for all p with $l(s, p_a) = 1$ as paths of length 1 imply that p_a is directly connected to s and therefore by definition of terminal edge capacities there is a corresponding edge from p_b to t. Let the pixels be ordered by the path length function and let q be the first pixel in this ordering for which $l(s, q_a) \neq l(t, q_b)$. Let p be the pixel just prior to q in the path from s to q_a. Let the node corresponding to p be p_b, implying $l(s, p_b) = l(s, q_a) - 1$. Since p occurs before q in the ordering, $l(t, p_a) = l(s, p_b) = l(s, q_a) - 1$ and there is an unsaturated path fragment from p_b to q_a. Lemma A.1, ensures that there will be an unsaturated path fragment from q_b to p_a as well. Since $l(t, p_a) = l(s, q_a) - 1$, therefore $l(t, q_b) = l(t, p_a) + 1 = l(s, q_a)$. We have handled the case of $l(s, q_a) < l(t, q_b)$. Other cases may be similarly proved. □

Lemma A.2 implies that for any pixel p if node p_a is reachable from s with path length l, then p_b is reachable from t with path length l as well. If in the residual graph p_b is also reachable from s, then there will be an augmenting path between s and t. This is not possible since the flow augmentation has stopped. Therefore both p_a and p_b can not be reachable from s.

B. Proof of Lemma 6.1

The proof of Lemma 6.1 relies on following observation;

Lemma B.1. *The submodular function $g^*(\cdot)$ created by AC can be transformed to an equivalent submodular function which has the same minimum but is monotone.*

Proof. We will focus our attention on a clique, say $\mathbf{c}$ on the cut and its derived fuction $g^*_c(\cdot)$. As explained function $g^*_c(\cdot)$ has been obtained by fixing the values of invalid constraints to be equal to the sum of effective flow in the cut edges covered by them. This makes the invalid constraints tight. AC has effectively found an optimal cut corresponding to $g^*(\cdot)$. However some of the values assigned to invalid constraints may be negative. The submodular function $g^*_c(\cdot)$ so created is therefore non monotone. We reparametrize the $g^*_c(\cdot)$ such that there are no negative valued constraints. Recall that reparametrization maintains submodularity and does not change the cut. Therefore, there will exist a flow corresponding to reparametrized $g^*_c(\cdot)$, which makes the same cut outputted by AC tight. Since constraints on the cut now are all non negative, effective flow in all the edges on the cut will also be non-negative (effective flow on the edges on the cut will be equal to the capacity of the edges which after reparametrization are non negative). A little reflection will show that that reparametrized $g_c^*(\cdot)$ is monotone.

□

In this discussion that follows we use $g^*(\cdot)$ to denote the function which results after reparametrization steps as explained above have been carried out.

Lemma B.2. *For the submodular function $g^*(\cdot)$ defined during the execution of AC and for all n-ary states x, y of a and b-gadgets the following is true:*

$$g^*(x, y) \geq g^*(x \cap \bar{y}, \bar{x} \cap y).$$

Proof. Let x and y be the n-ary states of a and b-gadgets respectively. Note that by the definition of $g^*(\cdot)$ the lemma holds for all feasible states for which $x = \bar{y}$. We need only to show it for infeasible states.

Due to Lemma B.1 we have for any two labeling (x, y) and (x', y'):

$$g^*((x, y) \cup (x', y')) \geq g^*((x, y) \cap (x', y'))$$

Using $x' = x \cap \bar{y}$ and $y' = \bar{x} \cap y$

$$g^*(x \cup (x \cap \bar{y}), y \cup (\bar{x} \cap y)) \geq g^*(x \cap (x \cap \bar{y}), y \cap (\bar{x} \cap y)))$$

Using simple identities like:

$$x = x \cup (x \cap \bar{y}),$$

$$y = y \cup (\bar{x} \cap y),$$

$$x \cap \bar{y} = x \cap (x \cap \bar{y}) \text{ and}$$

$$\bar{x} \cap y = y \cap (\bar{x} \cap y)$$

it can be shown that

$$g^*(x, y) \geq g^*(x \cap \bar{y}, \bar{x} \cap y)$$

Note that claim above is identical to Lemma 2 of [25], where max operator in the multi-label case is similar to union and min is similar to intersection in 2-label case. However, unlike [25], our proof based on the flow properties of AC does not require $g^*(\cdot)$ to be symmetric.

□

Labeling	Original	Rep. $x_3(b)$	Rep. $x_1(b)$
$D_{x_1}(a)$	20	20	20
$D_{x_1}(b)$	10	10	40
$D_{x_2}(a)$	10	10	10
$D_{x_2}(b)$	40	40	40
$D_{x_3}(a)$	20	20	20
$D_{x_3}(b)$	30	70	70

Table 1: Example Unary Potential

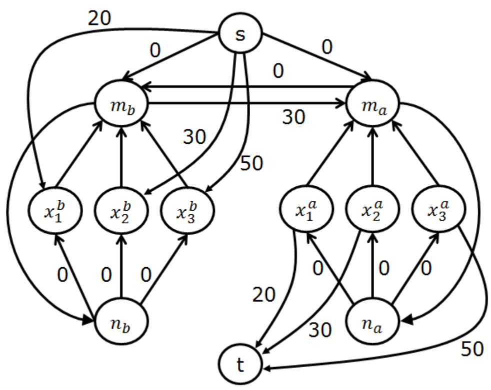

Figure 7: Flow graph for the example

C. Worked out example

Consider a 3 clique problem with nodes x_1, x_2 and x_3. Unary and clique potentials are as specified in first columns of Table 1 and 2 respectively. The second column of the two tables contains reparametrized energies after ball b in the well corresponding to pixel x_3 is raised by 40. The third column in both tables is the result of raising ball b in the well corresponding to pixel x_1 being raised by 30. These two reparametrization steps result in making the uniform labeling cost corresponding to bbb zero. Note that uniform labeling cost corresponding to aaa cannot be made zero as costs corresponding to states like abb are also zero after these two reparametrization steps and any further attempts to reduce a costs will make them negative;. Therefore the unary and clique potentials after reparametrization are as specified in third columns of respective tables.

The (a, b)-gadget based flow graph corresponding to the unary and clique potentials specified in column 3 of the two tables is given in Figure 7. Since $D_{x_1}(b) > D_{x_1}(a)$, $D_{x_2}(b) > D_{x_2}(a)$, and $D_{x_3}(b) > D_{x_3}(a)$ (Unary potential as given in Table 1 column 3) there are terminal edges from source s to all b-pixel nodes and terminal edges from all a-pixel nodes to sink t. Note that since the capacity of a pair of terminal edges (from s to $D_p(a)(D_p(b))$ and from $D_p(b)(D_p(a))$ to sink t) is defined to be equal to $|D_p(b) - D_p(a)|$ the capacity of terminal edges incident at a-pixel and b-pixel nodes corresponding to pixels x_1, x_2 and x_3 are 20, 30, and 50 respectively. In Figure 7 these are

Labeling	Original	Reparam. $x_3(b)$	Reparam. $x_1(b)$	First Augmentation Flow 20	Second Augmentation Flow 10
bbb	70	30	0	0	0
bba	30	30	0	0	0
bab	70	30	0	0	0
baa	50	50	20	0	0
abb	40	0	0	0	0
aba	30	30	30	30	20
aab	80	40	40	40	40
aaa	30	30	30	10	0

Table 2: Example Clique Potential

shown as labels of the terminal edges.

From source s there are also edges to auxiliary modes m_a and m_b. Capacity of the edge $s \to m_a$ is constrained by the fact that sum flows in edges $s \to m_a$ and $m_b \to m_a$ cannot exceed the cost of uniformly labeling the nodes of cliques as a. Since the reparametrized cost for uniformly labeling all nodes as a is 30, the capacity label of edges $s \to m_a$ and $m_b \to m_a$ is both 30. Similarly the capacity label of edges $s \to m_b$ and $m_a \to m_b$ is 0. The capacity of the two auxiliary edges $m_a \to n_a$ and $m_b \to n_b$ is infinity.

Since the flow graph corresponds to an image consisting of only a single 3-clique, the total number of DFCs (dual feasibility constraints) is 8. For the purposes of the following discussion and illustration we will denote a DFC only by it's labeling. The DFC aba is said to *cover/contain* $(n_a \to x_1^a, x_1^a \to m_a)$, $(n_b \to x_2^b, x_2^b \to m_b)$ and $(n_a \to x_3^a, x_3^a \to m_a)$ pairs of conjugate edges. Also, the conjugate edge pairs $(n_a \to x_1^a, x_1^a \to m_a)$, $(n_b \to x_2^b, x_2^b \to m_b)$ and $(n_a \to x_3^a, x_3^a \to m_a)$ are said to participate in the DFC aba. The initial allowed capacity of these three conjugate edge pairs due to the DFC aba is 30 (column 3 entry in row corresponding to aba in Table 2 which is the initial slack of the DFC in presence of zero flow or dual variables of type V having value zero). Since *residual capacity* of a conjugate edge pair is equal to the minimum of the slacks of all the DFCs in which it participates, initial residual capacity of conjugate edge pairs $(n_a \to x_1^a, x_1^a \to m_a)$, $(n_b \to x_2^b, x_2^b \to m_b)$ and $(n_a \to x_3^a, x_3^a \to m_a)$ is $0, 0$, and 0 respectively. This capacity is essentially the residual capacity of edges $n_a \to x_1^a$, $n_b \to x_2^b$ and $n_a \to x_3^a$. These pairs of conjugate edges can still carry flow. Flow can be pushed in $x_1^a \to m_a$, $x_2^b \to m_b$ and $x_3^a \to m_a$. The effect of this flow push in any one of these edges will be to create some slack in the DFC aba and to increase slack in other DFCs that also cover the edge. Since flow in these edges increases slack the effective residual capacity conjugate edges emanating from pixel nodes is infinity. Note that initial residual capacity of all conjugate edge pairs in the flow graph is 0.

An $s - t$ augmenting path in the residual graph is $s \to x_2^b \to m_b \to m_a \to n_a \to x_3^a \to t$ with residual capacities of the six edges being 30, $\inf$, 30, $\inf$, 0, and 50 respectively. Note that the edge $n_a \to x_3^a$ can be in the augmenting path because all DFCs that cover edges $n_a \to x_2^a$ and $n_a \to x_3^a$ (i.e. aaa and baa) have non zero slack. Amount of flow that can be augmented is actually 20 because DFC baa has prior to augmentation 20 slack. This slack becomes 0 when flow of 20 is sent through the chosen augmenting path. Slacks of all the DFCs after this augmentation are given in column four of Table 2.

The second $s - t$ augmenting path is $s \to x_1^b \to m_b \to m_a \to n_a \to x_3^a \to t$. Residual capacities of the edges are 20, $\inf$, 10, $\inf$, 0, 30. This time flow can be sent through $n_a \to x_3^a$ because all DFCs that cover $n_a \to x_1^a$ and $n_a \to x_3^a$ (aaa and aba) have slack. Second flow move sends a flow of 10 which is constrained by uniform labeling constraint aaa.

No further flow is possible and the the (S, T) cut formed at this stage is $(\{s, x_1^b, x_2^b, x_3^b, m_b, n_b\}, \{t, x_1^a, x_2^a, x_3^a, m_a, n_a\})$. AC outputs the primal labeling aaa. Primal unary cost is $20 + 10 + 20 = 50$ and clique potential for aaa is 30, making primal value 80. The value of the dual is equal to the flow sent in the flow graph, i.e., 30 plus the height of the lower ball in the wells, i.e., $20 + 20 + 10$, resulting in 80. AC for this example case gives optimal inference (even though the original clique potential is not submodular).