CS105L: Discrete Structures
I semester, 2006-07

Homework # 13

Due before the exam on **Saturday, December 2nd, 2006**

Instructor: Amitabha Bagchi

November 24, 2006

1. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals.

2. Let us consider a multigraph G. To avoid confusion let us invent the term *edge-holder* which means a vertex pair and let us use the term *edge* to mean an actual edge between two vertices. In a multigraph the set of edge-holders is a set, and the set of edges is a multiset chosen from the set of edge-holders. In fact if there is no edge corresponding to a given edge-holder, then that edge-holder is of no interest to us. With this is mind let us try to prove a generalization of Menger's theorem (the edge version.)

   Given a pair of vertices (s, t) in a graph G, let us consider a maximal set of edge disjoint paths P between s and t (note, the paths of P are not edge-holder disjoint.) We call such a set of paths *k-balanced* if every edge holder occurs in less than $|P|/k$ paths of P. If there is an edge holder which occurs exactly $\lfloor |P|/k \rfloor$ times in P then it is known as a *critical edge*.

   (a) Prove, using Menger's theorem, that if a set of edge-disjoint paths P between s and t is k-balanced then it is possible to find a set of k edge-holder disjoint paths between s and t using the edges in P.

   (b) Prove that it is possible to find $\lfloor |P|/k \rfloor$ sets of k-edge holder disjoint paths between s and t using the edges in P. Note that each set of k paths has to be edge holder disjoint within itself but might share edge holders with other such sets.

   To prove the second part you will need the following lemma:

   Lemma 13.1 *Given a k-balanced path system with critical edges between two vertices s and t, there is a set of k-edge holder disjoint paths from s to t which pass through every critical edge.*