

**Important:** The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a \* are optional challenge problems and are not to be discussed in the tutorial.

**Problem 1**

Suppose you have  $n$  men and  $n$  women and you have to seat them around a circular table so that men and women sit alternately. How many ways can you seat them?

**Problem 2**

Suppose there are  $n$  tennis players. How many ways can we pair them up so that everyone has a partner to play a singles match?

**Problem 3**

Given a set  $A$  of size  $m$  and set  $B$  of size  $n$  count the number of

1. Relations from  $A$  to  $B$ .
2. Total functions from  $A$  to  $B$ .
3. Partial functions from  $A$  to  $B$ .
4. Surjections from  $A$  to  $B$  (assume  $m \geq n$ ). These could be partial or total functions.
5. Injections from  $A$  to  $B$ . These could be partial or total. State whatever you assume about  $m$  and  $n$ .

**Problem 4 (Problem 15.7 of [1])**

For integers  $n, k \geq 0$  let  $S_{n,k}$  be the set of non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n.$$

**Problem 4.1**

Give a bijection between  $S_{n,k}$  and the set of binary strings with  $n$  zeroes and  $k$  ones. What is  $|S_{n,k}|$ ?

**Problem 4.2 ♠**

For integers  $n, k \geq 0$ , let

$$L_{n,k} := \{(y_1, \dots, y_k) \in \mathbb{N}^k : 0 \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Give a bijection from  $S_{n,k}$  to  $L_{n,k}$ .

**Problem 5**

Solve problem 15.8 of [1]. From there conclude that the number of trees with vertex set  $[n]$  is  $n^{n-2}$ .

**Problem 6**

Given a plane with integer points of the type  $(x, y)$  where both  $x$  and  $y$  are integers, we define a *lattice path* from  $(x_1, y_1)$  to  $(x_2, y_2)$  to be a set of line segments that go from a point  $(i, j)$  to  $(i + 1, j)$  or  $(i, j + 1)$ , i.e., all steps in the path either move right or up. Count the number of lattice paths between  $(0, 0)$  and  $(m, n)$ ?

**Problem 7**

We say that a function  $\pi$  is a *derangement of size  $n$*  if it is a bijection from  $\{1, \dots, n\}$  to itself (i.e., it is a permutation) and it has no fixed points, i.e.,  $\forall i : \pi(i) \neq i$ . Count the number of derangements of size  $n$ . The solution you submit must use Inclusion-Exclusion. Separately also try to see if you can solve the problem without the use of Inclusion-Exclusion.

**Problem 8**

Prove the following identities regarding binomial coefficients by making counting arguments. Give as many different arguments as possible

**Problem 8.1**

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}.$$

**Problem 8.2**

$$\binom{n}{k} \binom{n-k}{j} = \binom{n}{j} \binom{n-j}{k}.$$

**Problem 8.3**

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

**Problem 9 (Problem 15.47 of [1])**

Suppose  $n + 1$  numbers are selected from  $\{1, 2, \dots, 2n\}$  show using the Pigeonhole Principle that there must be two selected numbers whose quotient is a power of two.

**Problem 10 (Problem 15.50 of [1])**

Suppose  $2n + 1$  elements are selected from  $[4n]$ , use the Pigeonhole Principle to show that for every positive  $j$  that divides  $2n$  there must be two selected numbers whose difference is  $j$ .

**Problem 11**

Solve Problem 15.51 of [1].

**References**

- [1] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science June 2018, MIT Open Courseware.