COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 5: Trees.
15 September 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1
For $k \geq 2$ show that every k-connected graph contains a cycle of length $k + 1$.

Problem 2 *
Now show that if a k-connected graph has at least $2k$ vertices it also contains a cycle of length $2k$.

Problem 3 [1, Prob 20, page 31]
Show that every tree T has at least $\Delta(T)$ leaves.

Problem 4 ♠
Prove that if an acyclic graph has $n - k$ edges, it has k components.

Problem 5 [1, Prob 22, page 31]
Let F and F' be forests on the same vertex set with $|E(F)| < |E(F')|$. Show that F' has an edge e such that $F + e$ is also a forest.

Problem 6 [1, Corollary 1.5.4, page 15]
Prove that if T is a tree and G is any graph with $\delta(G) \geq |T| - 1$ then $T \subseteq G$, i.e., G has a subgraph isomorphic to T. Expand the proof idea given in the book into a proof.

Problem 7 [1, Prob 21, page 31]
Show that every tree without a vertex of degree 2 has more leaves than inner vertices. Show this by induction and then try to show it without induction.

Problem 8
Suppose a tree has the property that every internal vertex has degree at least 3. Show that there must be two leaves that have the same neighbour.

Problem 9
Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define their product graph $G_1 \times G_2 = (V, E)$ as follows: $V = V_1 \times V_2$ and $((x_1, y_1), (x_2, y_2)) \in E$ if $(x_1, x_2) \in E_1$ or $(y_1, y_2) \in E_2$. Prove or disprove the following statements:

1. The product of two regular graphs is regular. Recall a graph is called regular if all vertices have the same degree.
2. The product of two trees is a tree.
3. The product of two bipartite graphs is a bipartite graph.[1]

Problem 10 [1, Prob 25, page 31]
Prove by induction that every connected graph contains a normal spanning tree.

[1] A graph $G = (V, E)$ is called *bipartite* if there is a set $U \subseteq V$ such that for every $(u, v) \in E$, $u \in U$ and $v \in V \setminus U$.

Problem 11 [1, Prob 28, page 31]
Show that every automorphism of a tree fixes a vertex or an edge, i.e., for any one-to-one and onto function $f : V(T) \to V(T)$ that preserves the edge relationship for a tree T, either $f(v) = v$ for some $v \in V$ or there is an edge $(u, v) \in E(T)$ such that $f(u) = v$ and $f(v) = u$.

References

[1] Reinhard Diestel, *Graph Theory 5ed.*, Springer, 2016.