COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 7: Probability spaces and basic properties of probability measures
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Problem 1

Recall that in Quiz 7 we encountered the following setting: We have m balls and n bins. Each ball is thrown into one bin chosen at random. Let us define some notation

- $[k] = \{1, \ldots, k\}$, for $k \geq 1$.
- $B_i \in [n]$ denotes the bin into which ball i is dropped, $1 \leq i \leq m$.
- $S_i \subset [m]$ denotes the set of balls dropped into bin i, $1 \leq i \leq n$.

Problem 1.1

In this question you do *not* have to calculate any probability. We will describe some sets in words and you just have to write out mathematical descriptions of some sets in set builder notation using the notation introduced above.

1. The sample space Ω, i.e., the set of all possible outcomes.
2. Bin i is non-empty.
3. Bin i is empty.
4. No bin is empty.
5. No bin is non-empty.
6. Half the bins are empty. (You can assume n is even.)
7. At least half the bins are empty.
8. There is a bin with two balls in it.
9. No bin has more than k balls.
10. Bin i and Bin j have the same number of balls.
11. No two bins have the same number of balls.

Problem 1.2

Prove that if we throw the balls in such a way that $\Pr(B_i = j \cap B_k = \ell) = \Pr(B_i = j)\Pr(B_k = \ell)$, for all $i, k \in [m], j, \ell \in [n]$ then the resultant probability is the uniform probability distribution that we have assumed (i.e. each outcome is equally likely).

Problem 1.3

Now let us assume that every outcome in Ω is equally likely, i.e., each outcome has probability $1/|\Omega|$. Under this probability distribution try and calculate the probabilities of the events listed in Problem 1.1. **Warning**. No guarantee is being offered that these calculations are easy or even possible using normal methods. Explore the problems and solve them to the extent possible.

Problem 2
Recall that given a set of outcomes Ω, a σ-algebra is a collection $\mathcal{F} \subseteq 2^{\Omega}$ such that

- $\Omega \in \mathcal{F}$.
- $A \in \mathcal{F}$ implies $\Omega \setminus A \in \mathcal{F}$,
- $A_i \in \mathcal{F}$, $i \in \mathbb{N}$ implies $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$.

Problem 2.1
Given that Ω is finite, prove that any σ-algebra defined on Ω forms a complete lattice. Refer to Gallier's book for this problem.

Problem 2.2 *
If Ω is not finite, let's say it is countable, then is any σ-algebra defined on it a complete lattice? This is an exploratory question outside the scope of the course so don't waste time on it and don't ask your TAs for the solution.

Problem 2.3
Prove that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-algebras defined on the same set Ω then $\mathcal{F}_1 \cup \mathcal{F}_2$ is also a σ-algebra. Is this true for the intersection of a countable set of σ-algebras? Is the union of $\mathcal{F}_1$ and $\mathcal{F}_2$ also a σ-algebra? Prove it can or give a counterexample to show that it may not be.

Problem 3
In the previous tutorial sheet we studied the following setting. We call a coin *fair* if Pr(Heads) = PrTails = 1/2. We call the toss of a set of coins *independent* if the probability of the outcomes on any set of coins is simply the product of the probability of the individual outcomes. Let us consider a countably infinite sequence of independent tosses of this coin. So the set of outcomes is the (uncountable) set of all infinite sequences of T and H.

Let $\mathcal{F}_i$ be the σ-algebra corresponding to the first i coin tosses.

Problem 3.1
Show that $\mathcal{F}_i \subset \mathcal{F}_j$ for $j > i$. Note that you have to show it is a proper subset.

Problem 3.2
Revisit Problem 2.3 of the previous tutorial sheet in view of Problem 3.1 above. Here is the statement again: Argue *rigorously* but *without calculating any probability* that the probability of getting at least three heads when you flip five fair coins independently is $\leq$ the probability of getting at least three heads when you flip ten coins independently.

Problem 3.3
In Problem 2 we showed that the intersection of a countable set of σ-algebras is a σ-algebra. So we can say that $\mathcal{F} = \bigcap_{i \geq 1} \mathcal{F}_i$ is a σ-algebra. State in words three different events that belong in $\mathcal{F}$.

Problem 3.4 *
Try to calculate the probability of the events that you claimed were in $\mathcal{F}$ in Problem 3.3. Prove that all events in $\mathcal{F}$ have probability either 0 or 1? Note that this problem is exploratory and may be out of the scope of this course so don't ask your TAs for the solution.