COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 5: Partial orders
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Problem 1
Suppose we have a set S and a partially ordered set $(T, \preceq_T)$, let $\mathcal{F}$ be the set of functions $f : S \to T$, i.e., all the functions from S to T. We define a relation, $\preceq$, on $\mathcal{F}$ as follows: $f \preceq g$ if $f(x) \preceq_T g(x)$ for all $x \in S$. Show that $\preceq$ is a partial order on $\mathcal{F}$.

Problem 2
Two partially ordered sets $(S, \preceq_S)$ and $(T, \preceq_T)$ are said to be *isomorphic* if there exists a bijection $f : S \to T$ such that $x \preceq_S y$ if and only if $f(x) \preceq_T f(y)$ for all $x, y \in S$. The function f is called an *isomorphism.*

Also a function $f : S \to T$ is said to be *increasing* iff $x \preceq_S y$ implies $f(x) \preceq_T f(y)$. A function $f : S \to T$ is said to be *strictly* increasing iff $x \preceq_S y$ implies $f(x) \preceq_T f(y)$ and $f(x) \neq f(y)$ (this could also be denoted $f(x) \prec_T f(y)$).

With these definitions in hand attempt the following problems.

Problem 2.1
Suppose $(S, \preceq_S)$ and $(T, \preceq_T)$ are *isomorphic* and $f : S \to T$ is an isomorphism between them. Show that f and f^{-1} are both strictly increasing functions.

Problem 2.2
Suppose that $(S, \preceq)$ is a partially ordered set. Show that there exists an $\mathcal{S} \subseteq 2^S$ such that $(S, \preceq)$ is isomorphic to the partial order $(\mathcal{S}, \subseteq)$. (Is it clear that $(\mathcal{S}, \subseteq)$ is a partial order? Prove this first.)

Problem 2.3 *
Given a positive integer k let us consider the set $S = \{0, 1\}^k$ of all the vectors with k coordinates where each coordinate takes value either 0 or 1. Given two vectors $x, y \in S$ we say that $x \preceq y$ if $x_i \leq y_i$ for all $1 \leq i \leq k$ where x_i is the ith coordinate of x. Prove $\preceq$ is a partial order on S.

Given a function $f : S \to \mathbb{R}$ we define

$$\bar{f} = \frac{1}{2^k} \sum_{x \in S} f(x).$$

Now, suppose f and g are increasing functions from $(S, \preceq)$ to $(\mathbb{R}, \leq)$ (the real numbers partially ordered by the usual less than equal to relation), show that

$$\overline{fg} \geq \bar{f}\bar{g},$$

where the function fg at $x \in S$ is defined as $f(x)g(x)$.

Hint: Use induction on k. Note that this is an exploratory challenge problem so try to attempt it yourself and do not ask for your TA to help you.

Problem 3
Some basic properties of lattices.

Problem 3.1
Show Prop 4.2.2 of [Gallier08], i.e., show that if X is a lattice then the following identities hold for all $a, b, c \in X$

L1. $a \vee b = b \vee a$ and also $a \wedge b = b \wedge a$.

L2. $(a \vee b) \vee c = a \vee (b \vee c)$ and also $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

L3. $a \vee a = a$ and also $a \wedge a = a$.

L4. $(a \vee b) \wedge a = a$ and also $(a \wedge b) \vee a = a$.

Problem 3.2
A lattice is called *distributive* if, apart from the properties mention in Problem 3.1, we also have that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Give three examples of distributive lattices. Construct a non-distributive lattice.