COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 4: Part 1: Relations and infinite sets; Part 2: Pre-exam review
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Part 1: Relations and infinite sets

Problem 1
Prove that for any binary relations $\mathcal{R}$ and $\mathcal{S}$ on a set A,

1. $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$
2. $(\mathcal{R} \cap \mathcal{S})^{-1} = \mathcal{R}^{-1} \cap \mathcal{S}^{-1}$
3. $(\mathcal{R} \cup \mathcal{S})^{-1} = \mathcal{R}^{-1} \cup \mathcal{S}^{-1}$
4. $(\mathcal{R} \setminus \mathcal{S})^{-1} = \mathcal{R}^{-1} \setminus \mathcal{S}^{-1}$

Problem 2
Recall that in class we tried to prove that $\mathbb{N}^2$ is countable as follows. We defined the set $A_{2,3} = \{2^i \cdot 3^j : i, j > 0\}$ and constructed a mapping $f : \mathbb{N}^2 \to A_{2,3}$ such that for any $(i, j) \in \mathbb{N}^2, f(i, j) = 2^i 3^j$. First convince yourself that f is a bijection. Let us now complete the proof.

Problem 2.1
Recall the Schröder-Bernstein theorem: Given two sets A and B, if there is an injection from A to B and an injection from B to A there exists a bijection between A and B. To use this to complete the proof do the following

1. Define an injection from $A_{2,3}$ to $\mathbb{N}$.
2. Define an injection from $\mathbb{N}$ to $A_{2,3}$.
3. Prove that bijections compose, i.e., if f is a bijection between A and B and g is a bijection from B to C then $g \circ f$ is a bijection from A to C

After performing these three steps explain how they help us prove the result.

Problem 2.2
To avoid using Schröder-Bernstein's theorem we need to perform the "space-filling" argument to directly construct a bijection between $\mathbb{N}$ and $\mathbb{N}^2$. Write out explicitly the "space-filling" mapping for this purpose and demonstrate it is a bijection. Work out the generalisation of this mapping that gives us a bijection between $\mathbb{N}^d$ and $\mathbb{N}$ where $d \geq 2$.

Problem 3
Prove the following properties of countable sets. In doing so try to avoid using the Schröder-Bernstein theorem, i.e., try to construct bijections rather than two injections.

1. Every infinite subset of $\mathbb{N}$ is countable.
2. If A is a finite set and B is a countable set then $A \cup B$ is countable.
3. If A and B are countable sets then $A \cup B$ is also countable.

Part 2: Pre-exam review

All problems taken from [1].

Problem 4
In the Double Tower of Hanoi problem there are $2n$ disks of n different sizes, 2 of each size. As before we are to move all the disks from tower 1 to tower 3 using tower 2 for help, without placing a disk of (strictly) larger radius on top of a disk of (strictly) smaller radius. How many moves will it take to transfer the disks if disks of the same radius are indistinguishable from each other.

Problem 5
Solve the recurrence

$$g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0, \text{ for } n > 0,$$

with $g_0 = 1$. Try and solve it in multiple ways.

Problem 6
In the following assume that $A(x)$, $B(x)$ and $C(x)$ are the ordinary power series generating functions of the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ respectively. With this notation attempt the following problems:

Problem 6.1
If $c_n = \sum_{j+2k\leq n} a_j b_k$, express $C(x)$ in terms of $A(x)$ and $B(x)$.

Problem 6.2
If

$$nb_n = \sum_{k=0}^{n} 2^k \frac{a_k}{(n-k)!},$$

express $A(x)$ in terms of $B(x)$.

Problem 7
Solve the recurrence $g_0 = 0, g_1 = 1$ and

$$g_n = -2ng_{n-1} + \sum_{k=0}^{n} \binom{n}{k} g_k g_{n-k}, \text{ for } n > 1,$$

using an exponential generating function.

References

[1] R. L. Graham, D. E. Knuth, O. Patashnik. Concrete Mathematics: A foundation for computer science, 2nd ed. Pearson, 1994.