COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 12: Bipartite graphs, Euler tours and a return to counting.
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Problem 1
The height of a rooted tree is defined as follows: If a tree has a single vertex (the root) its height is 1, otherwise the height of the tree is 1 plus the maxium of the heights of the trees formed by the upward closure of the neighbours of the root. Prove that if a graph G has a normal spanning tree with height h then $\kappa(G) \leq 2h$, where $\kappa(G)$ is the connectivity of G, i.e., the minimum size of all vertex separators of G.

Problem 2
Let $G = (V, E)$ be a simple bipartite graph with bipartition $V = V_1 \cup V_2$. Let $n = |V|$, $r = |V_1|$, $s = |V_2|$ and $m = |E|$. Show that $m \leq rs$. Deduce that $m \leq n^2/4$ and describe the graphs where this holds at equality.

Problem 3 [1, Prob 31, page 32]
Prove or disprove that a graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertex.

Problem 4
Recall that given a graph $G = (V, E)$ a set of vertices $U \subseteq V$ is called an *independent set* if $(u, v) \notin E$ for all $u, v \in U$. Show that a graph G is bipartite if and only if every subgraph $G' = (V', E')$ of G has an independent set of size at least $V'/2$.

Problem 5
If W is a closed walk of length at least 1 that doesn't contain a cycle, show that some edge of W repeats itself immediately, i.e., if W is written as a sequence of edges $e_0, e_1, \ldots, e_k$ there must be an $0 \leq i \leq k-1$ such that $e_i = e_{i+1}$.

Problem 6 *
A labelled tree on n vertices is one in which each vertex is assigned a distinct label between 1 and n. Note that two isomorphic labelled trees are considered to be different. If t_n is the number of labelled trees on n vertices and f_n is the number of labelled forests on n vertices, show the following fact attributed to Pólya:

$$t_{n+1} = (n+1)f_n, (n \geq 0).$$

EGF Reloaded: You are on your own now.

Your TAs have been asked to not discuss any aspect of the remaining problems. Please tackle them independently. You can consider working in groups. If you decide you have spent enough time on a problem you may look it up on the Internet. In case there's any error in the writing, I apologise in advance, but I will not be entertaining any discussion on this part of the sheet.

Problem 7 (Indicators and EGFs)
A basic way of counting involves checking if a given set has a particular structure, with the "count" being 1 if the set satisfies that structural property and 0 otherwise. For example, the trivial structure

"Set" is satisfied by all sets, and so if we say f_n gives the value of the "count" for a set of size n, then $f_n = 1$, $n \geq 0$, which implies that the egf of f is e^x. Similarly if the structure is "Non-empty set" the corresponding egf is $e^x - 1$ since $f_0 = 0$. Write the egfs of the following structures

1. "1-element set."
2. "Empty set."
3. "Even-sized set."
4. "Odd-sized set."

Problem 8 (The *multiplication principle* for exponential generating functions)
Let g and h be types of discrete structures on sets of items, e.g., partitions, subsets, permutations, graphs, trees, etc. A $g \times h$-structure on a set A consists of

1. a partition A into disjoint subsets $A_1 \cup A_2 = A$,
2. a g-structure on A_1 and,
3. a h-structure on the set A_2,

where the structures on A_1 and A_2 are chosen independently.

Problem 8.1
Prove that if g-structures have egf $G(x)$ and h-structures have egf $H(x)$, show that $g \times h$ structures have egf $F(x) = G(x)H(x)$.

Problem 8.2
A binary tree is defined as a rooted tree in which the root is allowed to have at most 2 neighbours and each non-root vertex is allowed to have at most 3 neighbours. Using the multiplication principle of exponential generating functions prove that the number of labelled binary trees is $n!C_n$ where

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

is the n-th Catalan number. You may assume that the number of labelled binary trees on 0 vertices is 1 (the empty tree).

Problem 9 (The *composition principle* for exponential generating functions)
Let g and h be types of discrete structures on sets of items, e.g., partitions, subsets, permutations, graphs, trees, etc. Assume that h is the kind of structure such that if the set of items is empty then the number of h-structures that can be built on it is 0, e.g., there are 0 graphs on 0 vertices, so a graph satisfies the requirement on h, but if we take h to be the size of the power set of a set of items then the requirement is violated since the size of the power set of an empty set is 1. We define a *composite* $g \circ h$-structure on a set A as follows

1. Partition A into a set of blocks, B.
2. Independently create an h-structure in each block.
3. Create a g-structure on the set B.

An example of a composite structure can be found in Problem 9.1.

Problem 9.1
Show that every connected graph with edge connectivity 1 can be represented *uniquely* as a $g \circ h$ structure where an h-structure on a set A is a graph of edge connectivity at least 2 with vertex set A and a g-structure on a set A is a tree of size at least 2 with vertex set A. Note that if a I say my g-structure is a tree of size at least k with vertex set A, this definition is still valid for sets such that $|A| < k$, it's just that the *number* of trees of size at least k on such a set A is 0.

Problem 9.2
Given two kinds of structures, g-structures, with egf $G(x)$, and h-structures, with egf $H(x)$, if $H(0) = 0$, i.e. if the number of h-structures of an empty set is 0, then the egf of composite $g \circ h$-structures is given by $F(x) = G(H(x))$.

Problem 9.3
Write the structure "Partition" as a composition of the structures "Set" and "Non-empty set" and use the composition principle to derive the egf of the number of partitions of a set. Check your answer by looking up "Bell numbers". Can you derive the egf of the Bell numbers using the multiplication principle as well?

Problem 10 (Counting the number of rooted labelled trees on n vertices)
Prove Cayley's formula, i.e., prove that t_n, the number of rooted labelled trees on n vertices is n^{n-1}. There are several proofs but the one we want here goes as follows. If $T(x)$ is the egf of the sequence t_n, prove using the multiplication principle, the composition principle and, maybe, the result of Problem 6, that

$$T(x) = xe^{T(x)}.$$

In doing so remember to take the number of rooted labelled trees on 0 vertices as 0 otherwise the composition principle will not work. To solve this equation you will need the following theorem:

Theorem 1 (Lagrange Inversion Theorem)
Let $f(x), g(x)$ be power series such that $f(0) = g(0) = 0$ (i.e. the constant term is 0), and $f(g(x)) = x$, i.e., $f(x)$ and $g(x)$ are inverses of each other. Then

$$[x^n]g(x) = \frac{1}{n}[x^{-1}]\frac{1}{f(x)^n},$$

where $[x^i]f(x)$ is notation for the coefficient of x^i in the power series $f(x)$.

In particular, if $f(x) = x/\phi(x)$, $g(x) = x\phi(g(x))$, then

$$[x^n]g(x) = \frac{1}{n}[x^{n-1}]\phi(x)^n.$$

References

[1] Reinhard Diestel Graph Theory 5ed., Springer, 2016.