COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 11: Graph theory: Connectivity, Trees.
27 October 2017

Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Problem 1 (Menger's Theorem)
Given a graph $G = (V, E)$, two paths in G, $P_1 = (V_1 \subseteq V, E_1 \subseteq E)$ and $P_2 = (V_2 \subseteq V, E_2 \subseteq E)$ are called *edge disjoint* if $E_1 \cap E_2 = \emptyset$. Prove that G is ℓ-edge connected if and only if there are ℓ edge disjoint paths between u and v for every pair of vertices $u, v \in V$.

Problem 2
In Tutorial sheet 10 we came across this definition: Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$, i.e., V is the set of all 0-1 sequences of length d. We define the edge set as follows: there is an edge between two sequences if they differ in exactly one position. This graph is known as the d-dimensional cube. What is the connectivity of this graph, i.e., what is $\kappa(G)$? Consider the following two vertices: $x = 00\ldots0$ (all zeros) and $y = 11\ldots1$) (all ones vector). Find a set of vertex disjoint paths of *maximum size* between x and y.

Problem 3 (Expansion Lemma)
If G is a k-connected graph and G' is a graph obtained by adding by adding a new vertex u to G and placing edges between u and at least k vertices in G then G' is k-connected.

Problem 4 *
Suppose G is a simple graph with at least 3 vertices. Show that G is 2-connected if and only if for every triple of vertices x, y, z, there is an xz path through y.

Problem 5 [1, Prob 26, page 31]
Let G be a connected graph, and let r be a vertex of G. Starting from r move along the edges of G, going whenever possible to a vertex not visited so far. If there is no such vertex, go back along the edge though which the current vertex was first reached (unless the current vertex is r, in which case, stop). Show that the edges traversed form a normal spanning tree in G with r as the root. (This procedure has earned these trees the name *depth-first search trees*.)

Problem 6 [1, Prob 21, page 31]
Show that a tree without any vertex of degree 2 has more leaves than non-leaf vertices. Can you find a short proof that doesn't use induction?

References

[1] Reinhard Diestel Graph Theory 5ed., Springer, 2016.