COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 10: Graph theory basics
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet.

Problem 1
Prove Theorem 1.5.1 of [1], i.e. show that the following statements are equivalent

1. $T = (V, E)$ is a tree (i.e. T is a connected, acyclic–meaning it has no cycles–undirected simple graph with no self-loops);
2. Any two vertices of T are connected by a unique path in T;
3. T is minimally connected, i.e., T is connected but $T - e$ is disconnected for every $e \in E$. (Recall that the notation $T - e$ refers to the subgraph of T given by $(V, E \setminus \{e\})$, where $e \in E$);
4. T is maximally acyclic, i.e., T is acyclic but $T + (x, y)$ has a cycle where x and y are any two non-adjacent vertices of T.

Problem 2 [1, Prob 2, page 30]
Let $d \in \mathbb{N}$ and $V = \{0, 1\}^d$, i.e., V is the set of all 0-1 sequences of length d. We define the edge set as follows: there is an edge between two sequences if they differ in exactly one position. This graph is known as the d-dimensional cube. Determine the average degree, diameter, girth and circumference of the *d-dimensional cube*. Note that the circumference of a graph is the length of the longest cycle in the graph.

Problem 3 [1, Prob 3, page 30]
Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

Problem 4 [1, Prob 9, page 30]*
$G = (V, E)$ is said to be a *connected* graph if there is a path in G connecting each $u, v \in V$. Show that every connected graph contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

Problem 5 [1, Prob 22, page 31]
Let F, F' be forests on the same set of vertices with $\|F\| \leq \|F'\|$. Show that F' has an edge e such that $F + e$ is also a forest.

Problem 6 [1, Prob 27, page 31]
Let $\mathcal{T}$ be a collection of subtrees of a tree T. Recall that a subtree is a subgraph which is a tree.

1. Show that if the trees in $\mathcal{T}$ have pairwise non-empty intersection then their overall intersection is non-empty. (To better appreciate this property of trees, give an example of a graph G and a collection of subgraphs such that pairwise intersection of the subgraphs does *not* imply overall intersection. What is it in the nature of trees that makes this property true for them but not for general graphs? AB.)
2. For $k \in \mathbb{N}$, show that either $\mathcal{T}$ contains k disjoint trees or there is a set of $k - 1$ vertices that meets (i.e. is part of) every tree in $\mathcal{T}$. (Clearly this will be true only for values of k related to the size of T. But try to make the argument general so that it works if the sizes of both T and $\mathcal{T}$ are *not* finite. AB.)

Problem 7 [1, Prob 28, page 31]
Show that every automorphism of a tree fixes either a vertex or an edge, i.e., if $\phi : V \to V$ is an automorphism of a tree $T = (V, E)$ then either there is a $u \in V$ such that $\phi(u) = u$ or there is a $(u, v) \in E$ such that $(\phi(u), \phi(v)) \in E$. (In case you are confused, note that it is possible for the second case to occur without the first case occuring. AB.)

References

[1] Reinhard Diestel Graph Theory 5ed., Springer, 2016.