COL202: Discrete Mathematical Structures. I semester, 2017-18.
Minor II
8 October 2017
Maximum Marks: 15

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Important: Keep your answers within the boxes prescribed for each question. Anything written outside the box will be treated as rough work. Solve the problem first on the separate rough sheets provided then copy carefully into the printed sheet. **Rough sheets will not be collected.**

Problem 1 (2 marks)
Let $X = \sum_{i=1}^{n} X_i$ where $\{X_i\}_{i=1}^{n}$ is a mutually independent collection of identical random variables. Each X_i takes value -1 with probability 1/2 and 1 with probability half. Use Markov's inequality with a suitable transformation of X to show that $\mathrm{P}\{X > a\} \leq e^{-a^2/4n}$. (Hint: $\sum_{i\geq 0} x^{2i}/(2i)! \leq \sum_{i\geq 0} x^{2i}/i! = e^{x^2}$. Note: If you find a constant different from 4 in the denominator of the exponential on the RHS, that is okay, it doesn't mean the problem is wrong.)

Problem 2 (1 mark)
We are given k independent random bits $r_1, r_2, \ldots, r_k$. We define a function $f : 2^{[k]} \rightarrow \{0, 1\}$ as follows: $f(S) = \bigoplus_{i\in S} r_i$, i.e. $f(S)$ is the XOR of all the random bits whose index is in S. (Recall $a \oplus b = 0$ if a and b are the same and 1 otherwise.) Prove that $\{f(S) : S \subseteq 2^{[k]}\}$ is a pairwise independent collection of random variables.

Problem 3 (1+1 = 2 marks)
Two events, A and B, are said to be *negatively correlated* if $\mathrm{P}\{A \cap B\} \leq \mathrm{P}\{A\}\mathrm{P}\{B\}$.

Problem 3.1 (1 mark)
Show that if $A \cup B = \Omega$ then A and B are negatively correlated events.

Problem 3.2 (1 mark)
Is it possible for two events A and B to be independent if $A \cup B = \Omega$ and $0 < \mathrm{P}\{A\}, \mathrm{P}\{B\} < 1$? Either give an example to show that it is, or argue using the solution of Problem 3.1 that it is not.

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Problem 4 (1+1 = 2 marks)
Given two partially ordered sets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, we said in class that a (total) function $f : X \to Y$ is *monotone* if $x \preceq_X y$ implies $f(x) \preceq_Y f(y)$ for all $x, y \in X$. Now we say that f is an *order embedding* if the converse also holds, i.e., f is monotone and $f(x) \preceq_Y f(y)$ implies $x \preceq_Y y$. If f is an order embedding that is onto Y (i.e. $f(X) = Y$) then f is call an *order isomorphism*. If an order isomorphism exists between two partially ordered sets then they are said to be *order isomorphic*.

Problem 4.1 (1 mark)
Show that an order embedding must be one-to-one.

Problem 4.2 (1 mark)
Give an example of a one-to-one monotone map that is not an order embedding. If you feel that all one-to-one monotone maps are order embeddings you may try to write a proof for that.

Problem 5 (2+1+2=5 marks)
Given a partially ordered set $(X, \preceq)$, a set $A \subseteq X$ is known as an *up set* of $(X, \preceq)$ if $x \in A$ and $x \preceq y \in X$ implies that $y \in A$, i.e., if an element x of X is in A then *all* elements that are related to x and are greater than it are in A.

We derive another partially ordered set from $(X, \preceq)$ as follows: Let $\mathcal{U}(X)$ be the collection of the up sets of X with the usual partial order on sets, $\subseteq$. You may find the following notion useful: Given $A \subset X$, $\uparrow A = A \cup \{y \notin A : \exists x \in A : x \preceq y\}$ is the smallest up set containing A.

Problem 5.1 (2 marks)
Show that $(\mathcal{U}(X), \subseteq)$, derived from $(X, \preceq)$ is a lattice.

Problem 5.2 (1 mark)
A lattice is said to be *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z in the lattice. Is $(\mathcal{U}(X), \subseteq)$ a distributive lattice?

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Problem 5.3 (2 marks)
Recall that given two partially ordered sets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, the *product order*, $\preceq_{X\times Y}$ on $X \times Y$ is defined as follows $(x_1, y_1) \preceq_{X\times Y} (x_2, y_2)$ if *both* $x_1 \preceq_X x_2$ and $y_1 \preceq_Y y_2$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Below when we refer to a product of partially ordered sets it should be understood that we are talking about the product of the two sets ordered by the product partial order.

Given two *disjoint* partially ordered sets $(X, \preceq_X)$ and $(Y, \preceq_Y)$ show that the partially ordered set $\mathcal{U}(X \cup Y)$ is order isomorphic to $\mathcal{U}(X) \times \mathcal{U}(Y)$.

Problem 6 (1 + 3 = 4 marks)
We are given the space $(\{0,1\}^k, \mathcal{F} = 2^{\{0,1\}^k})$, i.e., each outcome is a bit string of length k and the σ-algebra is the power set of the outcome space. If we define random variable $X_i(\omega)$ as the ith coordinate of the outcome ω, $1 \leq i \leq k$, we will work with a probability function $\mathrm{P}\{\cdot\}$ that ensures that the collection of random variables $\{X_i : 1 \leq i \leq k\}$ is *mutually independent.* In class we discussed that the *uniform* probability measure on this space has this property, so we will assume $\mathrm{P}\{\cdot\}$ is the uniform measure.

First, note that $(\mathcal{F} = 2^{\{0,1\}^k}, \leq_k)$ is a partially ordered set which is the k-way product of the partially ordered set $(\{0,1\}, \leq)$ where $\leq$ is the usual order on integers (c.f. Problem 5.3 for definition of product orders). Now, consider two events, $A, B \in \mathcal{F}$, which are up sets of the partially ordered set $(\mathcal{F}, \leq_k)$ (see definition of up set in Problem 5). Prove by induction on k that these two events are *positively correlated*, i.e.,

$$\mathrm{P}\{A \cup B\} \geq \mathrm{P}\{A\}\mathrm{P}\{B\}.$$

Problem 6.1 (1 mark)
Base case: $k = 1$

Problem 6.2 (2 marks)
Induction hypothesis and step: