

TUTORIAL SHEET 8

We shall use $N(\mu, \sigma)$ to denote the normal distribution with mean μ and variance σ^2 . We shall use $E(\lambda)$ to denote the exponential distribution whose density $f(x)$ is as follows: $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$; 0 otherwise.

1. Let X be a random variable whose distribution is given by $N(0, 1)$ and $t > 0$. Show that

$$\Pr[X \geq t] \leq \frac{1}{\sqrt{2\pi t}} e^{-t^2/2}.$$

You may want to use the fact that if $x \geq t$, then $e^{-t^2/2} \leq \frac{x}{t} e^{-t^2/2}$.

2. Heights of men in a city follow the distribution $N(170, 9)$. Use the previous exercise to upper bound the proportion taller than 175 cm. Use Chebyshev's inequality to bound this proportion and compare the two values.
3. Let X_1, \dots, X_{50} be identical and independent random variables, each with mean 4 and variance 20. Let $Y = X_1 + \dots + X_{50}$.
 - Estimate the distribution of Y using the central limit theorem and use this to estimate $\Pr[Y > 120]$.
 - Use Chebyshev bound to estimate this probability and compare the two bounds.
4. You arrive at a bus station and let X denote the waiting time for the next bus. Assume that X follows the distribution $E(\lambda)$. Show that

$$\Pr[X > t + s | X > s] = \Pr[X > t].$$

Interpret this property in words.

5. Let X and Y be two independent random variables with uniform distribution on $[0, 1]$. What is $\Pr[X + Y \geq 1]$?
6. Let X_1, \dots, X_n be independent random variables, where X_i has distribution $E(\lambda_i)$. What is the distribution of $\min(X_1, \dots, X_n)$?
7. Let X and Y be two independent random variables with density f_1 and f_2 respectively. Show that the density $g(z)$ of the random variable $Z = X + Y$ is given by

$$g(z) = \int_{-\infty}^{\infty} f_1(x) f_2(z - x) dx.$$

- Let X and Y have distribution $E(\lambda_1)$ and $E(\lambda_2)$. What is the density of $X + Y$?
- Show that if X and Y have distribution $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$, then $X + Y$ has distribution $N(\mu_1 + \mu_2, \sigma_1 + \sigma_2)$.