COL 7004 October 21, 23, 25

TUTORIAL SHEET 11

1. Let $v_1, \ldots, v_r$ be r orthonormal vectors. Show that they are linearly independent.

2. Let v_1, v_2 be two independent vectors in $\Re^n$. Show that there are two orthonormal vectors w_1, w_2 such that $\mathsf{span}(w_1, w_2) = \mathsf{span}(v_1, v_2)$ and $\mathsf{span}(w_1) = \mathsf{span}(v_1)$. Generalize this result to n independent vectors $v_1, \ldots, v_n$.

3. Let $v_1, \ldots, v_\ell$ be orthonormal vectors in $\Re^n$. Let V be the subspace spanned by these vectors. Let x be an arbitrary vector in $\Re^n$. Show that the vector $v \in V$ which minimizes $||v - x||$ is given by
$$x = \sum_{i=1}^{\ell} \langle v_i, x \rangle v_i.$$
You may want to use the following steps:
   (i) Show that if $\langle a, b \rangle = 0$, then $||a - b||^2 = ||a||^2 + ||b||^2$.
   (ii) Show that if x is defined as above and w is any other vector in V, then $\langle v - w, x - w \rangle = 0$.

4. Let V be a subspace of $\Re^n$. Let W be the set of vectors $\{x : \langle x, v \rangle = 0, \text{ for all } v \in V\}$. Show that W is a subspace and $W \cap V = \{0\}$. Show that every vector $x \in \Re^n$ can be uniquely written as $x_1 + x_2$ where $x_1 \in V, x_2 \in W$.

5. Let U, V, W be three vectors spaces with basis B_U, B_V, B_W respectively. Let T be a linear transformation from U to V such that its matrix with respect to these bases is given by M, and similarly let T' be a linear transformation from V to W whose matrix is M'. Define a linear transformation T'' from U to W as follows: $T''(u) = T'(T(u))$. Show that the matrix of T'' with respect to bases B_U and B_W is given by $M \cdot M'$.