COL 7004 October 13, 14, 16

TUTORIAL SHEET 10

1. Are the vectors $(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0), (2, 1, 1, 6)$ linearly independent? Find a basis for the subspace spanned by these vectors.

2. Prove that if a set of vectors $A = \{v_1, \ldots, v_n\}$ span a vector space V, then a maximal indepedent subset of these vectors is a basis of V. A subset A' of A is said to be maximal independent if the vectors in A are independent and for any vector $v_i \in A \setminus A'$, the vectors in $A' \cup \{v_i\}$ are not linearly independent.

3. Show that the vectors $(1, 0, -1), (1, 1, 1), (1, 0, 0)$ form a basis of $\Re^3$. What are the coordinates of the vector (a, b, c) in this basis?

4. Let W_1 and W_2 be subspaces of a vector space V. Let $W_1 + W_2$ denote the set of vectors: $\{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$. Show that $W_1 + W_2$ is also a subspace of V and its dimension is at most the sum of the dimensions of W_1 and W_2.

5. Let V be the vector space of all 2×2 real matrices. What is the dimension of this vector space? Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$$

Show that W_1, W_2 are subspaces of V. Find the dimensions of $W_1, W_2, W_1 \cap W_2, W_1 + W_2$.

6. Let W be a subspace of a vector space V. Let dimension of W be k and dimension of V be n. Let $w_1, \ldots, w_k$ be a basis of W. Then show that there are vectors $v_{k+1}, \ldots, v_n \in V$ such that $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$ is a basis of V.

7. Let W_1 and W_2 be two subspaces of a vector space V. Show that $d(W_1) + d(W_2) = d(W_1 \cap W_2) + d(W_1 \cup W_2)$. Here $d(W)$ denotes the dimension of a subspace W of V. [Hint: start with a basis of $W_1 \cap W_2$ and extend it to basis of W_1 and to a basis of W_2.]

8. Let $v_1, \ldots, v_k$ be a set of vectors in $\Re^n$ and V denote the span of these vectors. Let A be an $m \times n$ matrix. Let V' denote the span of $Av_1, \ldots, Av_k$. Show that the dimension of V' is at most the dimension of V. Show that the two dimensions are equal if A has rank n.

9. Let the columns of a matrix A be $A_1, \ldots, A_k$. Define a new matrix A' whose columns are same as that of A except that the second column $A'_2 = A_2 - cA_1$, where c is a real number. Show $A' = AJ$, where J is the matrix which is same as the identity matrix along with $J_{12} = -c$. What is the inverse of J?

10. Let A be a square matrix. We say that A is invertible if there is a matrix B such that $AB = I$. Show that if A is invertible iff columns of A are linearly independent. Show that if $AB' = I$ for a matrix B', then $B = B'$.

11. Show that if A is an $n \times k$ matrix and B is a square inverible matrix, then the rank of A and AB are same. Show that the dimension of the nullspace of A and AB are same.

12. Let $A = \{e_1, \ldots, e_n\}$, $B = \{f_1, \ldots, f_n\}$, $C = \{g_1, \ldots, g_n\}$ be three sets of basis of a vector space V of dimension n. Let P be the $n \times n$ matrix, where the i^{th} column of P denotes the coordinates of f_i in the basis A. Similarly, let Q be the $n \times n$ matrix, where the i^{th} column of Q denotes the coordinates of g_i in the basis B. Show that the i^{th} column of PQ denotes the coordinates of g_i in the basis A. Use this fact to show that the matrix P (and similarly the matrix Q) is invertible.