COL 754

Homework II

Due on 28th September, 2018

All questions are from the textbook.

1. In the maximum cut problem, we are given as input an undirected graph $G = (V, E)$ with nonnegative weights $w_{ij} \geq 0$ for all $(i, j) \in E$. We wish to partition the vertex set into two parts U and $W = V - U$ so as to maximize the weight of the edges whose two endpoints are in different parts. We will also assume that we are given an integer $k \leq |V|/2$, and we must find a partition such that $|U| = k$.

   - Show that the following nonlinear integer program models the maximum cut problem with a constraint on the size of the parts:

   $$\begin{aligned} \max . \sum_{(i,j)\in E} & w_{ij}(x_i + x_j - 2x_i x_j) \\ \sum_{i\in V} x_i &= k \\ x_i &\in \{0, 1\} \quad \forall i \in V \end{aligned}$$

   - Show that the following linear program is a relaxation of the problem:

   $$\begin{aligned} \max . \sum_{(i,j)\in E} & w_{ij} z_{ij} \\ z_{ij} &\leq x_i + x_j \quad \forall (i, i) \in E \\ z_{ij} &\leq 2 - x_i - x_j \quad \forall (i, j) \in E \\ \sum_{i\in V} x_i &= k \\ 0 \leq z_{ij} &\leq 1 \quad \forall (i, j) \in E \\ 0 \leq x_i &\leq 1 \quad \forall i \in V \end{aligned}$$

   - Let $F(x) = \sum_{(i,j)\in E} w_{ij}(x_i + x_j - 2x_i x_j)$ be the objective function from the nonlinear program. Show that for any (x, z) that is a feasible solution to the linear programming relaxation, $F(x) \geq 1/2 \cdot \sum_{(i,j)\in E} w_{ij} z_{ij}$.
   - Argue that given a fractional solution x, for two fractional variables x_i and x_j , it is possible to increase one by $\varepsilon > 0$ and decrease the other by ε such that $F(x)$ does not decrease and one of the two variables becomes integer.
   - Use the arguments above to devise a 1/2-approximation algorithm for the maximum cut problem with a constraint on the size of the parts.

2. In the maximum directed cut problem (sometimes called MAX DICUT) we are given as input a directed graph $G = (V, A)$. Each directed arc $(i, j) \in A$ has nonnegative weight $w_i j \geq 0$. The goal is to partition V into two sets U and $W = V - U$ so as to maximize the total weight of the arcs going from U to W (that is, arcs (i, j) with $i \in U$ and $j \in W$). Consider the following linear programming relaxation for this problem:

$$\max . \sum_{(i,j)\in A} w_{ij} z_{ij}$$
$$z_{ij} \leq x_i \quad \forall (i,j) \in A$$
$$z_{ij} \leq 1 - x_j \quad \forall (i,j) \in A$$
$$0 \leq z_{ij} \leq 1 \quad \forall (i,j) \in E$$
$$0 \leq x_i \leq 1 \quad \forall i \in V$$

Consider a randomized rounding algorithm for the maximum directed cut problem that solves a linear programming relaxation of the integer program and puts vertex $i \in U$ with probability $1/4 + x_i/2$. Show that this gives a randomized 1/2- approximation algorithm for the maximum directed cut problem.

3. In the maximum coverage problem, we are given a set of elements E, and m subsets of elements $S_1, \ldots, S_m \subseteq E$ with a nonnegative weight $w_j \geq 0$ for each subset S_j. We would like to find a subset $S \subseteq E$ of size k that maximizes the total weight of the subsets covered by S, where S covers S_j if $S \cap S_j \neq \emptyset$.

   - Show that the following nonlinear integer program models the maximum coverage problem:

$$\max . \sum_{j\in[m]} w_j \left(1 - \prod_{e\in S_j} (1 - x_e)\right)$$
$$\sum_{e\in S_j} x_e = k$$
$$x_e \in \{0,1\} \quad \forall e \in E$$

   - Show that the following linear program is a relaxation of the maximum coverage problem:

$$\max . \sum_{j\in[m]} w_j z_j$$
$$\sum_{e\in S_j} x_e \geq z_j \quad \forall j \in [m]$$
$$\sum_{e\in S_j} x_e = k$$
$$0 \leq z_j \leq 1 \quad \forall j \in [m]$$
$$0 \leq x_e \leq 1 \quad \forall e \in E$$

- Using the technique from Question 1 above, give an algorithm that deterministically rounds the optimal LP solution to an integer solution and has approximation ratio of $1 - 1/e$.

4. In the capacitated dial-a-ride problem, we are given a metric (V, d), a vehicle of capacity C, a starting point $r \in V$, and k source-sink pairs $s_i - t_i$ for $i = 1, ..., k$, where $s_i, t_i \in V$. At each source s_i there is an item that must be delivered to the sink t_i by the vehicle. The vehicle can carry at most C items at a time. The goal is to find the shortest possible tour for the vehicle that starts at r, delivers each item from its source to its destination without exceeding the vehicle capacity, then returns to r; note that such a tour may visit a node of V multiple times. We assume that the vehicle is allowed to temporarily leave items at any node in V.

    - Suppose that the metric (V, d) is a tree metric (V, T). Give a 2-approximation algorithm for this case.
    - Give a randomized $O(\log |V|)$-approximation algorithm for the capacitated dial-aride problem in the general case.