Computer Graphics

Subodh Kumar

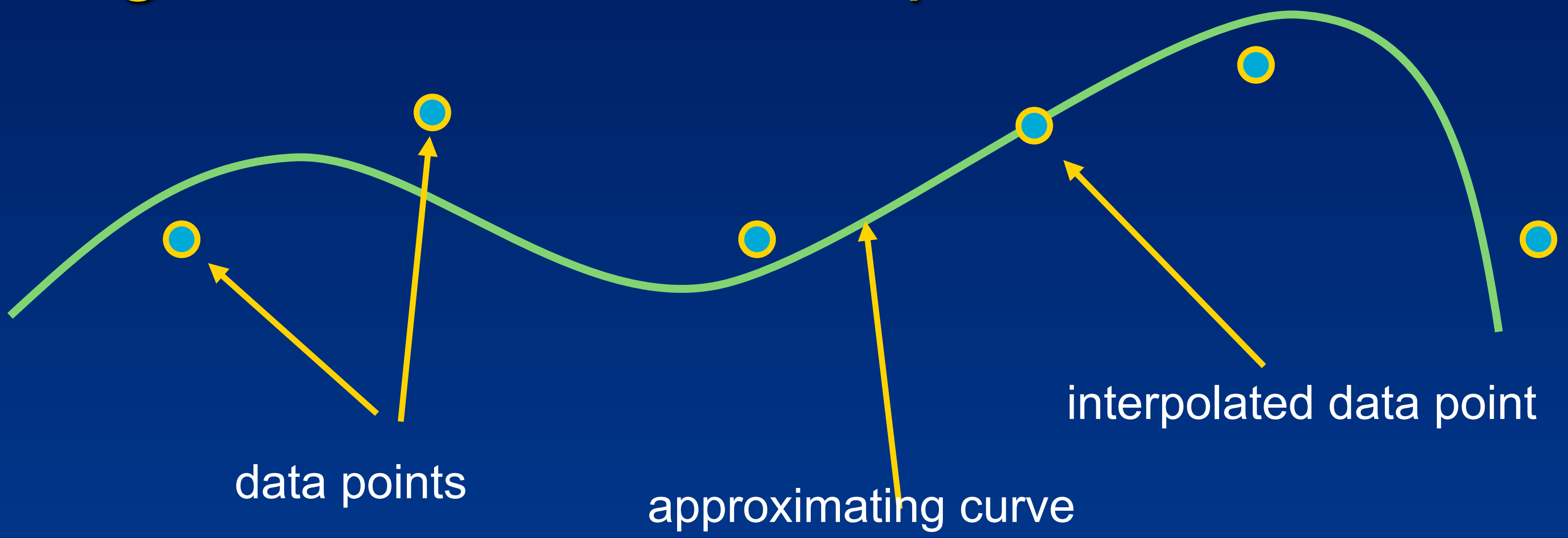
Dept of Computer Sc. & Engg.
IIT Delhi

Curves

- Analytic Function
  - Parametric form: *p(t)*
  - Implicit form: F(t) = 0
- Design is based on "data points"

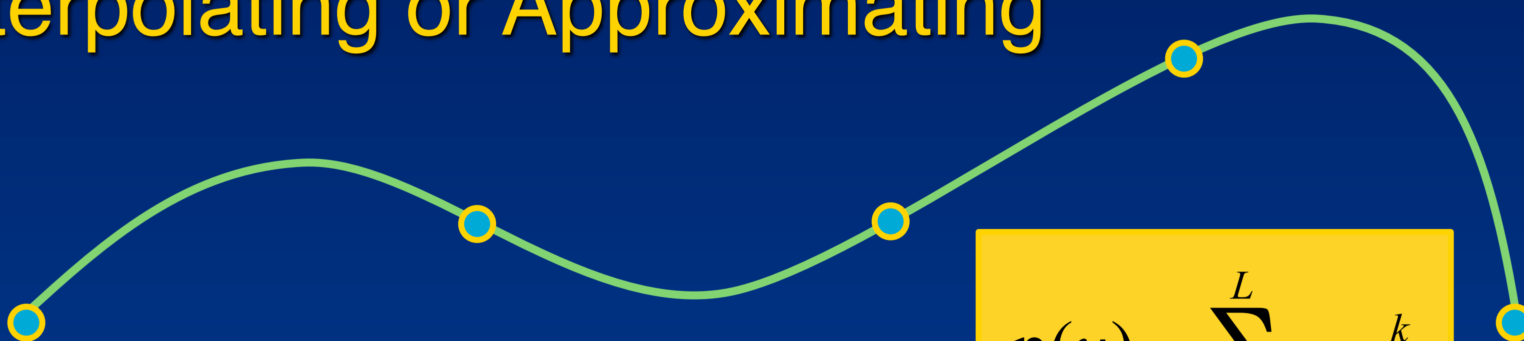
Curves

- Analytic Function
  - Parametric form: *p(t)*
  - Implicit form: F(t) = 0
- Design is based on "data points"



Representation

- Stable
- Smooth
- Easy to evaluate
  - including derivatives etc.
- Interpolating or Approximating



$$p(u) = \sum_{k=0}^{L} c_k u^k$$

Polynomial

Matrix Form

Consider cubic curve:

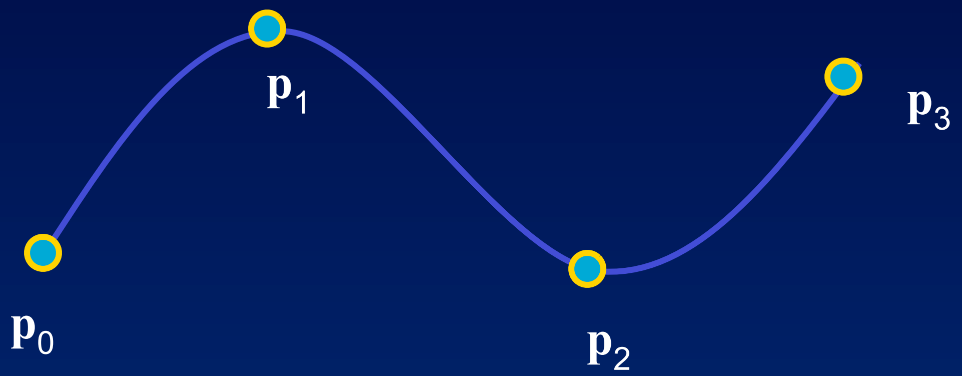
$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$$

$$\mathrm{p}(u) = \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$$

Interpolating Curve

Find cubic $\mathbf{p}(u)$ interpolating $\mathbf{p}_i$: Find $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$



$$\mathbf{p} = \mathbf{A}\mathbf{c}$$

Apply interpolation at u=0, 1/3, 2/3, 1

$p_0 = p(0) = c_0$

$p_1 = p(1/3) = c_0 + (1/3)c_1 + (1/3)^2 c_2 + (1/3)^3 c_2$

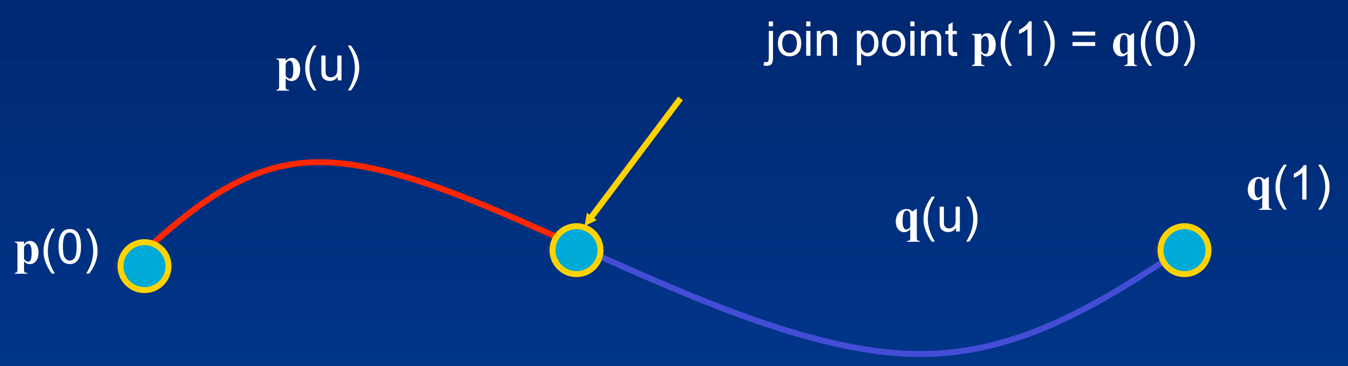
$p_2 = p(2/3) = c_0 + (2/3)c_1 + (2/3)^2 c_2 + (2/3)^3 c_2$

$p_3 = p(1) = c_0 + c_1 + c_2 + c_2$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

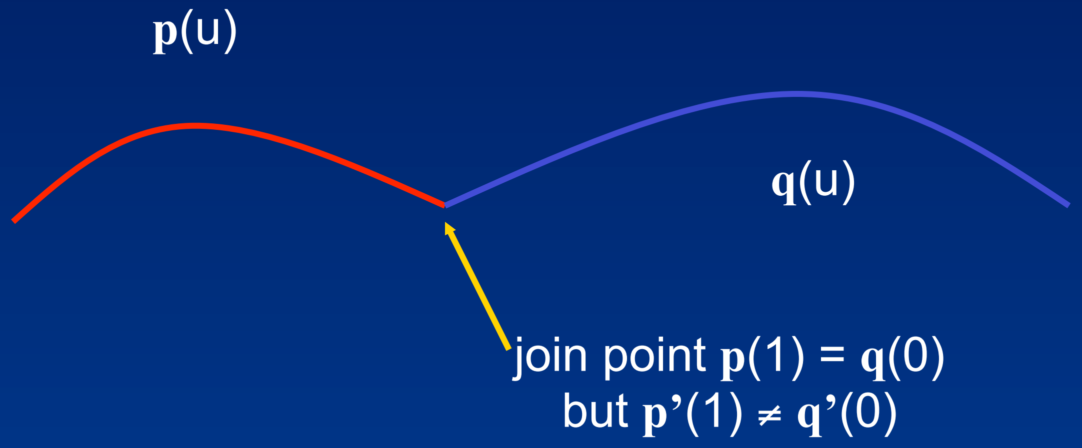
Doesn’t work well for higher degrees!

Spline: Curve Segments

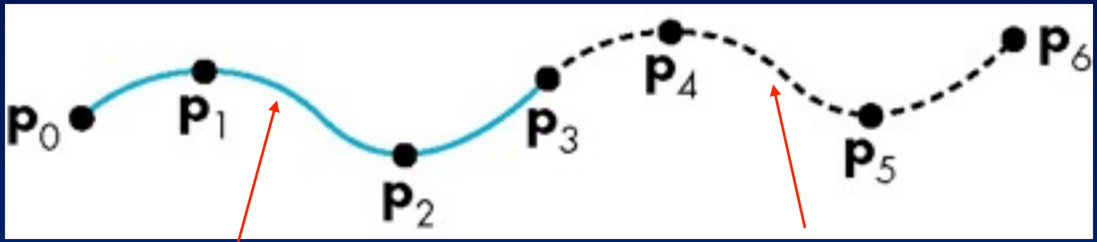


Polynomial Splines

- Easy to evaluate
- Continuous and differentiable everywhere
  - Must worry about continuity at join points including continuity of derivatives



Multi-Segment Interpolation



use **p** = $[p_0\ p_1\ p_2\ p_3]^T$

use **p** = $[p_3\ p_4\ p_5\ p_6]^T$

Independent interpolation in groups of 4?

Blending Functions

Or, basis functions:

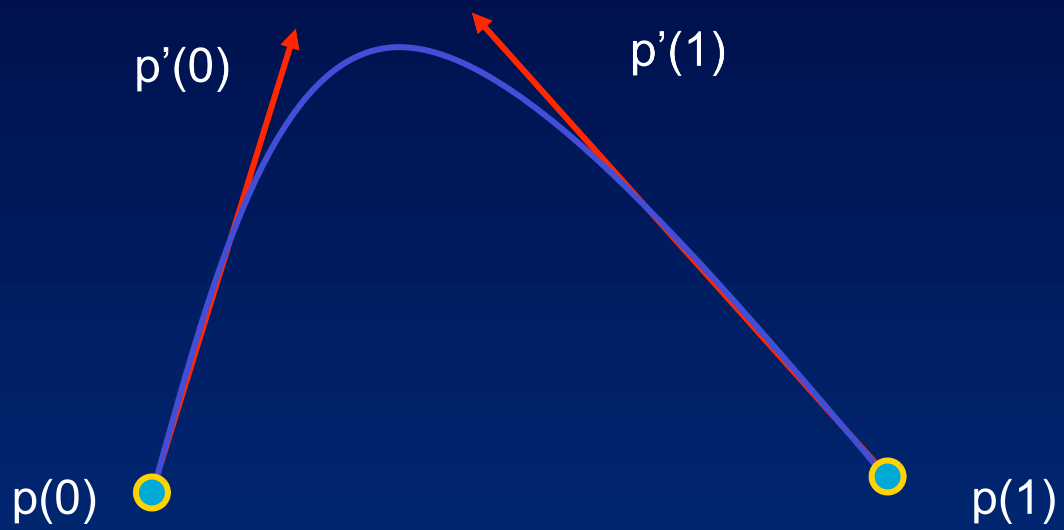
$$p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$$

Example:

$$b_0(u) = -4.5(u-1/3)(u-2/3)(u-1)$$
$$b_1(u) = 13.5u\ (u-2/3)(u-1)$$
$$b_2(u) = -13.5u\ (u-1/3)(u-1)$$
$$b_3(u) = 4.5u\ (u-1/3)(u-2/3)$$

If the basis functions are not smooth, the interpolation isn't either

Hermite Form



Use two interpolating conditions and two derivative conditions per segment

Ensures continuity and first derivative continuity between segments

Equations

$p(u) = c_0 + uc_1 + u^2c_2 + u^3c_3$

Hence:

$p(0) = p_0 = c_0$

$p(1) = p_3 = c_0 + c_1 + c_2 + c_3$

And: $p'(u) = c_1 + 2uc_2 + 3u^2c_3$

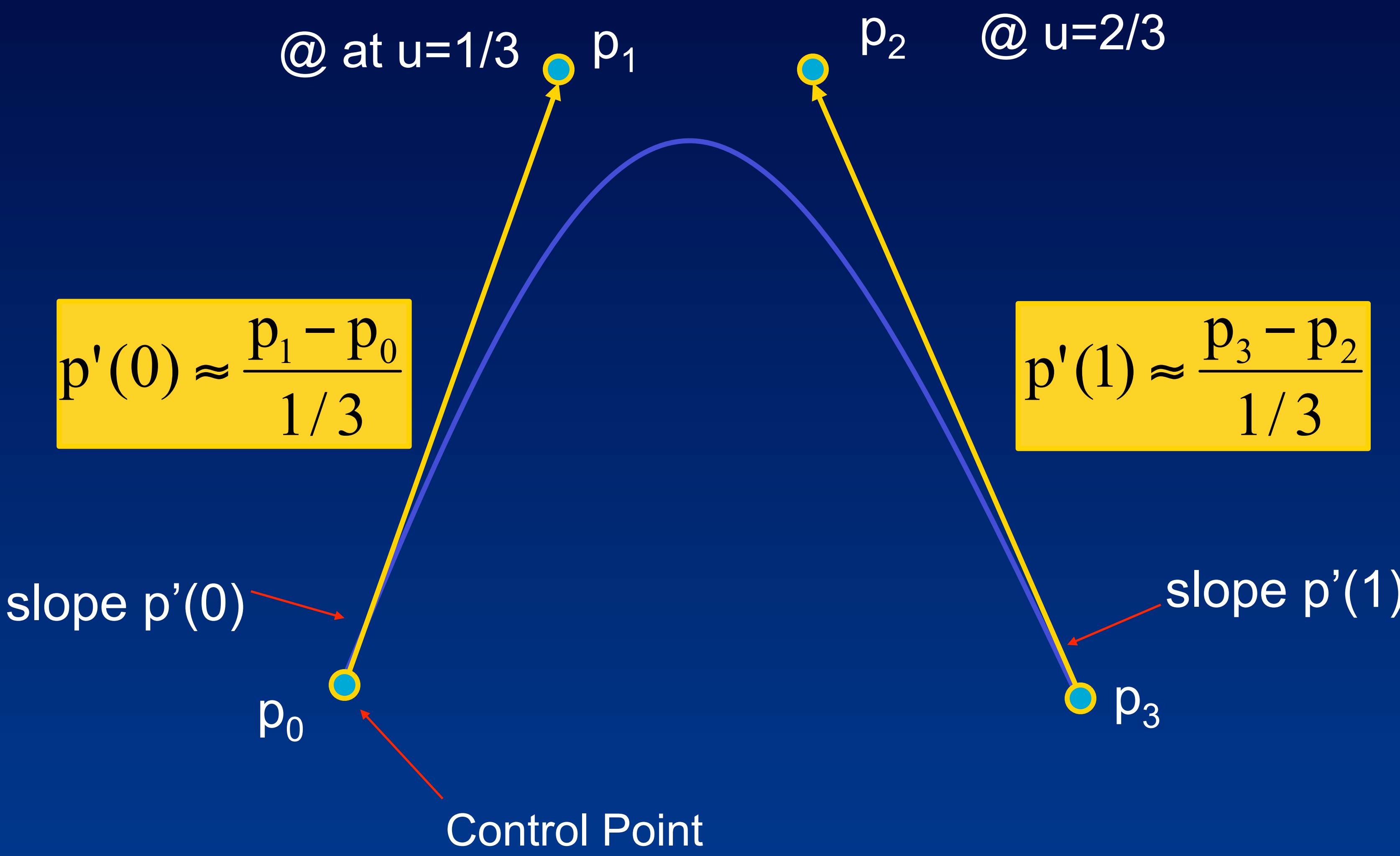
Hence:

$p'(0) = p'_0 = c_1$

$p'(1) = p'_3 = c_1 + 2c_2 + 3c_3$

But derivatives must be given!

Approximating Derivatives



$$p'(0) \approx \frac{p_1 - p_0}{1/3}$$

$$p'(1) \approx \frac{p_3 - p_2}{1/3}$$

Equations

Now

$p'(0) = 3(p_1 - p_0) = c_0$

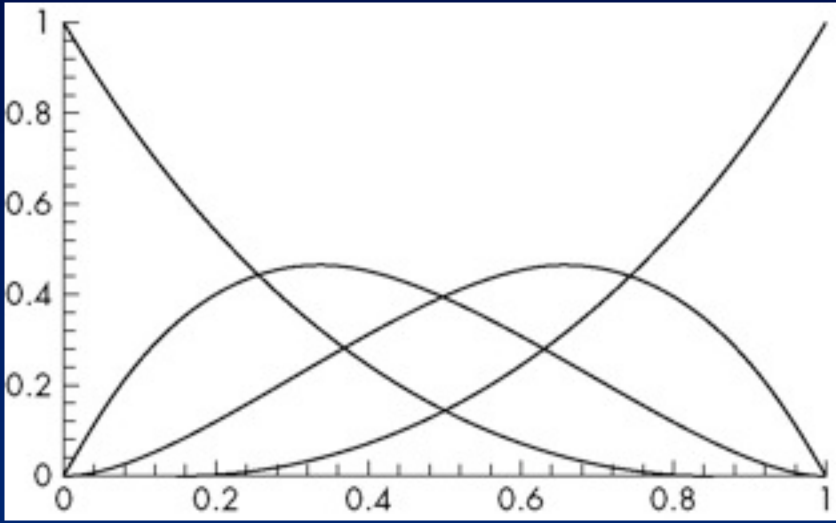
$p'(1) = 3(p_3 - p_2) = c_1 + 2c_2 + 3c_3$

$$\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$p(u) = \mathbf{u}^T \mathbf{M}_B \mathbf{p} = \mathbf{b}(u)^T \mathbf{p}$

Blending Functions

$$\mathbf{b}(u) = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}$$



Bernstein Polynomials

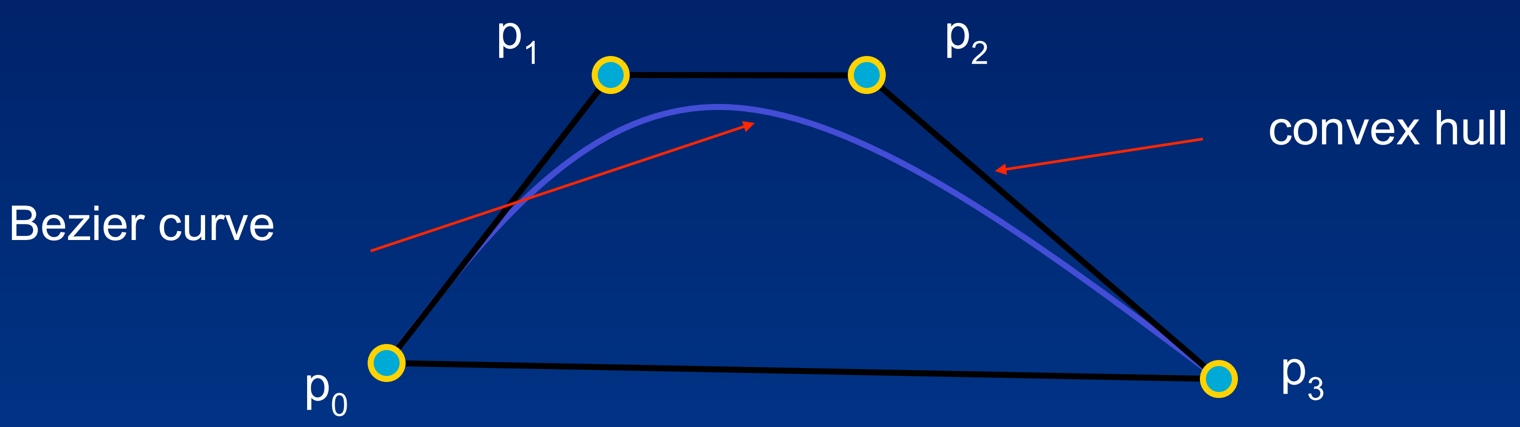
- The blending functions:

$$b_{\mathrm{kd}}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

- All zeros at 0 and 1
- For any degree they all sum to 1
- They are all between 0 and 1 inside (0,1)

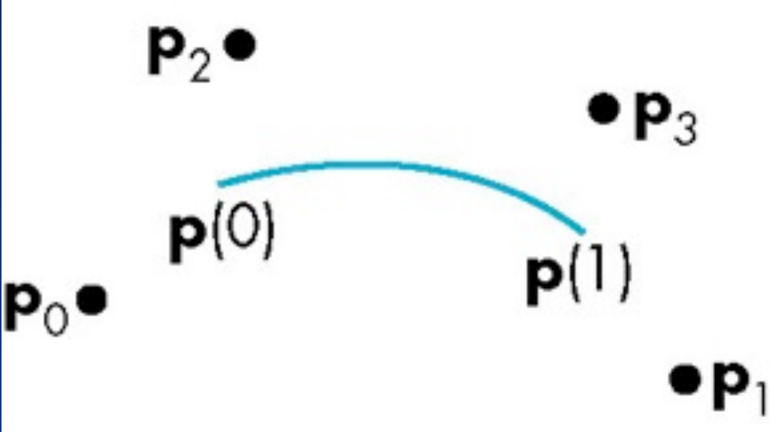
Convex Hull Property

- Bezier curves lie in the convex hull of their control points
- Non-interpolating



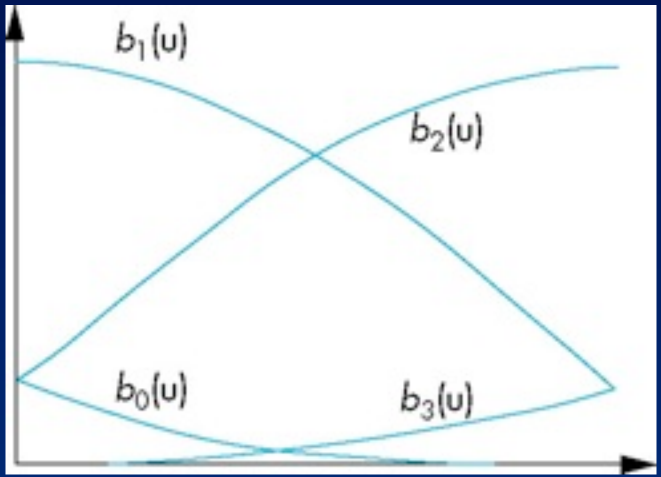
B-Splines

- Basis splines: $p = [p_{i-2}\ p_{i-1}\ p_i\ p_{i-1}]T$
  defines curve only between p_{i-1} and p_i
  - Local control
- More continuity equations at joints
  - e.g., cubics can provide continuity of function, the first derivative and the second derivative



Blending Functions

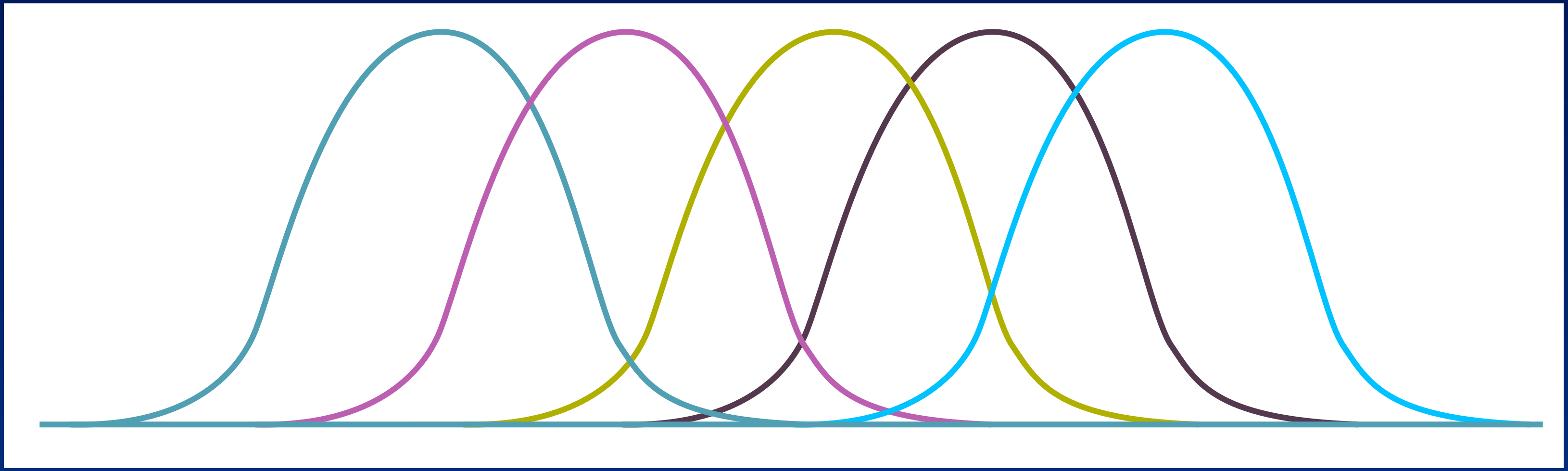
$$\mathbf{b}(u) = \frac{1}{6}\begin{bmatrix} (1-u)^3 \\ 4-6u^2+3u^3 \\ 1+3u+3u^2-3u^2 \\ u^3 \end{bmatrix}$$



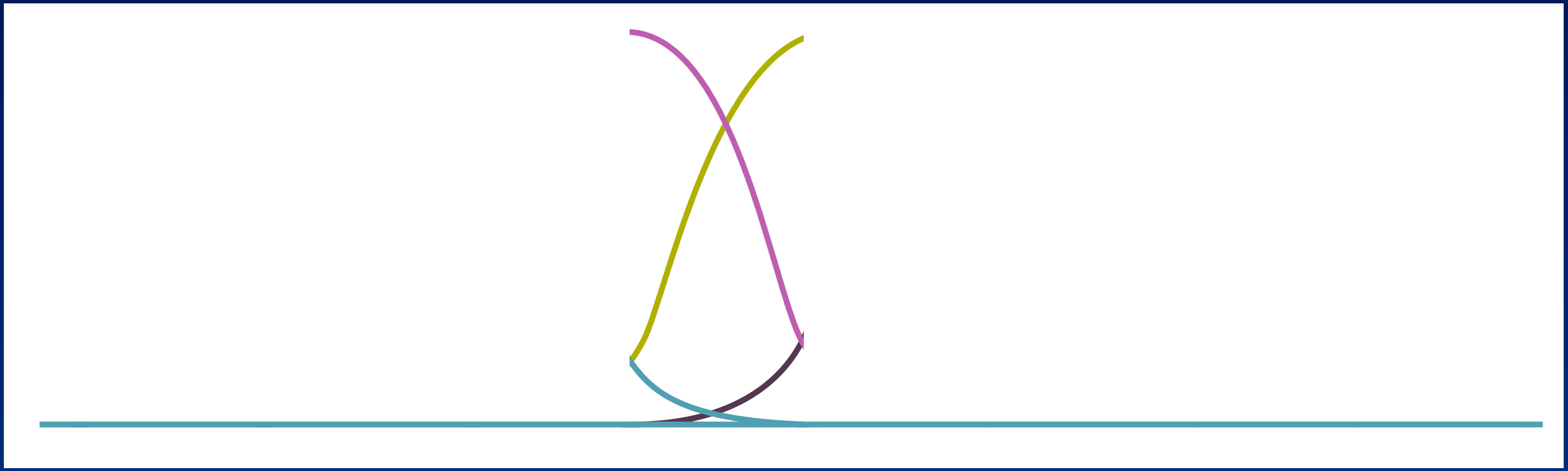
$p(u) = \mathbf{u}^T\mathbf{M}_S\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$

$$\mathbf{M}_S = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

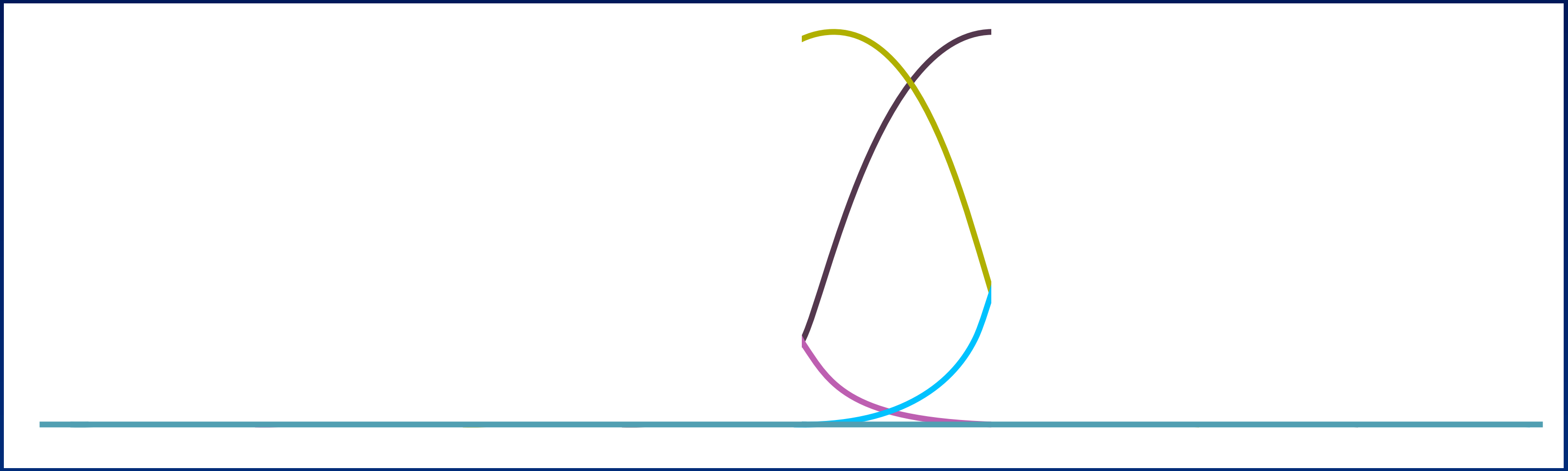
Shifted Bases



Shifted Bases



Shifted Bases

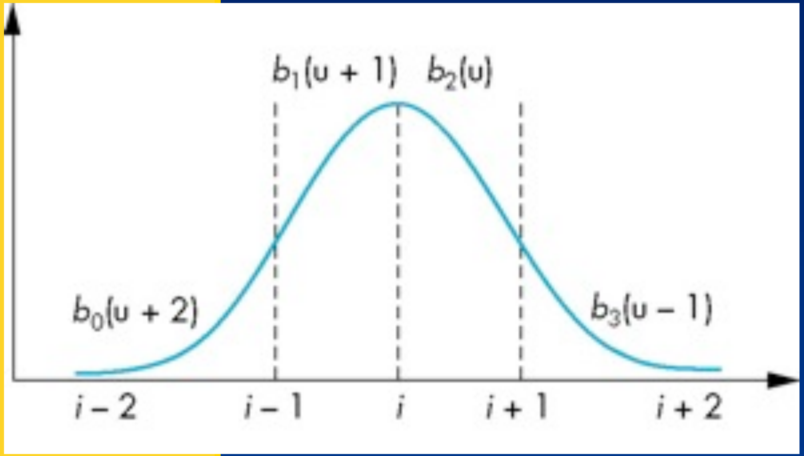


Splines and Basis

- Interior control point control four segments

$$p(u) = \sum B_i(u)\, p_i$$

$$B_i(u) = \begin{cases} 0 & u < i-2 \\ b_0(u+2) & i-2 \le u < i-1 \\ b_1(u+1) & i-1 \le u < i \\ b_2(u) & i \le u < i+1 \\ b_3(u-1) & i+1 \le u < i+2 \\ 0 & u \ge i+2 \end{cases}$$



General Splines

- Any degree
- Many bases
- Interpolating or approximating
- Data at arbitrary parameter values (knots)
  - Even repeated knots allowed
- Rational splines