

CSL361 Problem set 8: Householder and Givens Orthogonalization

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1 Householder QR

Let $v \in \mathbb{R}^n$ be non-zero. The $n \times n$ matrix (rank 1 corrections to the identity)

$$P = I - 2 \frac{vv^T}{v^T v}$$

is called the *Householder reflection*. Show that:

1. $P = P^t = P^{-1}$.
2. When a vector $x \in \mathbb{R}^n$ is multiplied by P , it is reflected across the hyperplane $\text{span}\{v\}^\perp$.

3. If

$$v = x \pm \|x\|_2 e_1$$

Then,

$$Px = \mp \|x\|_2 e_1 \in \text{span}\{e_1\}$$

That is, a *Householder reflection* along $v = x \pm \|x\|_2 e_1$ annihilates all but the first component of x while preserving its norm.

4. Argue that out of the two choices for v for annihilating all but the first component of x , $v = x + \text{sign}(x_1)\|x\|_2 e_1$ is better from a numerical point of view. Explain the two choices geometrically.
5. If $P = I - 2vv^T/v^T v$ and A is a matrix, then show that

$$PA = \left(I - 2 \frac{vv^T}{v^T v} \right) A = A + vw^t$$

where $w = \beta A^t v$ and $\beta = -2/v^t v$. Thus, a Householder update of a matrix involves a matrix-vector multiplication followed by a outer-product update. Argue that treating P as a general matrix increases work by an order of magnitude.

Also show that the above produces a Householder vector \hat{v} near to the exact v . That is if $\hat{P} = I - 2\hat{v}\hat{v}^T/\hat{v}^T\hat{v}$, then

- (a) $\|\hat{P} - P\|_2 = O(\mu)$
- (b) $fl(\hat{P}A) = P(A + E)$, where $\|E\|_2 = O(\mu\|A\|_2)$

6. More generally, for a given m -vector x , consider the partitioning

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x_1 is a $(k - 1)$ -vector, $1 \leq k < m$. Show that if we take the Householder vector to be

$$v = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} - \alpha e_k$$

where $\alpha = -\text{sign}(x_k)\|x_2\|_2$ (x_k being the k^{th} component of x), then the resulting Householder transformation annihilates the last $m - k$ components of x .

7. Using the above results show that a sequence of Householder transformations can be defined for $k = 1, \dots, n$ such that all the subdiagonal entries of matrix are annihilated. Show that such a sequence defines a QR factorization.

2 Givens rotations

Householder reflections are useful for introducing zeros on a grand scale. However, in situations where it is necessary to zero elements more selectively, Givens rotations are useful. These are rank 2 corrections to identity of the

form

$$G(i, k, \theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

where $c = \cos \theta$ and $s = \sin \theta$ for some θ .

1. Show that Givens rotations are orthogonal. In particular, show that premultiplication by $G(i, k, \theta)^t$ amounts to a counterclockwise rotation by θ radians in the (i, k) coordinate plane.
2. Show that k^{th} component of a vector can be set to 0 by premultiplication with $G(i, k, \theta)^t$ where

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}} \quad s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}}$$

Show also that only rows i and k are affected.

3. Describe a procedure for Given's QR .
4. Show that the numerical properties of Givens QR are as favourable as Householder QR .