

CSL361 Problem set 3: Polynomials and interpolation

August 19, 2009

1 First some more rounding and chopping

1. Consider the numbers

$$\begin{aligned}x1 &= 0.1234 \times 10^1 \\x2 &= 0.3429 \times 10^0 \\x3 &= 0.1289 \times 10^{-1} \\x4 &= 0.9895 \times 10^{-3} \\x5 &= 0.9763 \times 10^{-5}\end{aligned}$$

Add these numbers using four-decimal-digit chopped floating point arithmetic in both forward and reverse. Which is more accurate? Why?

2. Suggest methods for evaluating each of

$$\begin{aligned}\text{(a)} \quad e^x &\simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \text{(b)} \quad \cos x &\simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\ \text{(c)} \quad \sin x &\simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\end{aligned}$$

for $x = 25$ up to 12 digits of accuracy. Pay attention to both truncation and round-off errors. Try'em out with Matlab programs.

3. Suppose you have to evaluate

$$F(x) = x \sin x / (1 - \cos x)$$

near $x = 0$. Do you foresee any problems? Suggest a method to overcome the problem. Again try out on Matlab.

2 Interpolation

1. Derive the *divided difference* formula again (not to be done in the tute class).
2. Consider the following algorithm for computing $d_k = f[x_1, \dots, x_k]$, $k = 1, 2, \dots, n + 1$ given the data points x_1, x_2, \dots, x_n and function values f_1, f_2, \dots, f_n .

Algorithm:

```
Input  $\{x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n\}$   
for  $j = 1, 2, \dots, n + 1$   
     $d_j \leftarrow f_j$   
    for  $k = 1, 2, \dots, n$   
        for  $j = n + 1, n, \dots, k + 1$   
             $d_j \leftarrow (d_j - d_{j-1}) / (x_j - x_{j-k})$   
Output  $\{d_1, d_2, \dots, d_{n+1}\}$ 
```

Prove the correctness and determine the time and space requirements.

3. Prove again the following result done in the class (not in the tute class).

If $p(x)$ is the interpolating polynomial which agrees with $f(x)$ at $n + 1$ points in $[a, b]$. If f is $n + 1$ times continuously differentiable in $a, b]$ the for any $\bar{x} \in [a, b]$ there is a value $\eta \in [a, b]$ such that the *truncation error* is given by

$$E_T(\bar{x}) = f(\bar{x}) - p(\bar{x}) = [f^{(n+1)}(\eta) / (n+1)!] (\bar{x} - x_1) (\bar{x} - x_2) \dots (\bar{x} - x_{n+1})$$

What conclusion can you draw about extrapolation/truncation errors?

4. Show that under the conditions of the previous problem, and with P_k the polynomial that interpolates f at x_1, x_2, \dots, x_{k+1} , for $k = 1, 2, \dots, n$, the difference

$$P_{k+1}(\bar{x}) - P_k(\bar{x})$$

is an estimate of the truncation error in $P_k(\bar{x})$. This estimate is usable whenever $f^{(k+1)}(x)$ does not change greatly in the interval containing x_1, \dots, x_{k+2} and \bar{x} .

5. Consider Horner's method for evaluating a polynomial

```

Input  $\{n, a_0, a_1, \dots, a_n\}$ 
Input  $x$ 
 $p \leftarrow a_n$ 
for  $k = n - 1, n - 2, \dots, 0$ 
     $p \leftarrow x * p + a_k$ 
Output  $\{x, p\}$ 

```

Show that the **backward error estimate** is given by

$$\hat{p} = \hat{a}_n x^n + \dots + \hat{a}_1 x + \hat{a}_0$$

where,

$$\hat{a}_k = \begin{cases} a_k \langle 2n \rangle & k = n \\ a_k \langle 2k + 1 \rangle & k = n - 1, \dots, 0 \end{cases}$$

Conclude that if $nr\mu \leq 0.1$, then the relative error bound can be obtained as

$$\frac{|\hat{a}_k - a_k|}{|a_k|} \leq \begin{cases} 2n\mu' & k = n \\ (2k + 1)\mu' & k = n - 1, \dots, 0 \end{cases}$$

Also, show that the forward error estimate is given by

$$|\hat{p} - p(x)| = [2n |a_n x^n| + (2n - 1) |a_{n-1} x^{n-1}| + 3 |a_1 x| + |a_0|] \mu'$$

6. Given Algorithm in 2 for computing the divided difference coefficients, an algorithm for evaluating Newton's formula can be given as

```

Input  $\{x_1, \dots, x_{n+1}, d_1, \dots, d_{n+1}\}$ 
Input  $x$ 
 $p \leftarrow d_{n+1}$ 
for  $i = n, n - 1, \dots, 1$ 
     $p \leftarrow p * (x - x_i) + d_i$ 
Output  $\{x, p\}$ 

```

Show that the Algorithms in 2 and 6 evaluate a polynomial \hat{p} that interpolates a function \hat{f} at x_1, x_2, \dots, x_{n+1} . For all x in an interval of length l that contain x_1, x_2, \dots, x_{n+1} , the following bound holds:

$$|f(x) - \hat{f}(x)| \leq 9(n^3 + n^2 + 1)(l^n/m^n) \bar{f}\mu'$$

where,

$$\bar{f} = \max\{|f(x_i)| : i = 1, 2, \dots, n + 1\}$$

and

$$m = \min\{|x_i - x_j| : i, j = 1, 2, \dots, n + 1, i \neq j\}$$

[**Note:** This one will be hard! Perhaps you should split it up into several sub-problems?]

7. Conclude from the previous problems that the rounding errors in the two algorithms are equivalent to a change in the function f . This change will be small provided that
 - (a) n is not too large
 - (b) the interval length is not large
 - (c) the data points x_1, x_2, \dots, x_{n+1} are not too close together, and
 - (d) the maximum value \bar{f} is not much larger than any other $f(x)$ for x in the interval.

8. Suppose that each of the following functions has been tabulated at $x_1 = 0, x_2 = 0.1, x_3 = 2$ and $x_4 = 0.3$. Estimate the truncation error at $\bar{x} = 0.15$ for the polynomial of degree 3 that interpolates at the given tabulated points.
 - (a) $f(x) = \sin x$
 - (b) $f(x) = 2x^3$
 - (c) $f(x) = 1/(x + 1)$

9. Use Algorithm 6 to approximate $f(-0.5)$ where f is the function $f(x) = 1/(1 + 25x^2)$. Print out a table of $(n, f(-0.5), \epsilon)$ (ϵ being the absolute error) for $n = 1, 2, 3, 4$. Also print out the truncation error estimates $P_{n+1} - P_n$. Verify that the derivatives of this function change so rapidly that these error estimates are not so accurate.

10. Suppose f is a function that has been tabulated at $x = 0, 0.1, 0.2, 0.3, 0.4, \dots$ and suppose $|f^{(k)}(x)| \leq k, k = 0, 1, 2, \dots$. What is the best value of n to use for interpolating f at $\bar{x} = 0.25$? That is, what value of n will result in the sum of truncation error and rounding error to be as small as possible?