Approximation Algorithms for some Important Geometric Problems

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Outline

1. Travelling Salesman Problem
2. Rectangle Stabbing Problem

Travelling Salesman Problem

Problem

Input: An undirected graph $G = (V, E)$, with each edge $e \in E$ attached with an integer cost $w(e) > 0$.

Objective: Find a Hamiltonian cycle of G with minimum cost.

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Status of the problem: The decision version is **NP-complete.**

Δ-TSP

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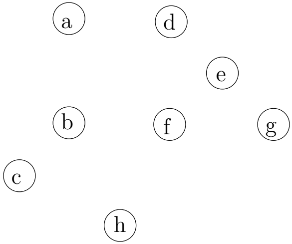
i.e., for all triple of vertices $u, v, w \in V$, we have $w(u, v) + w(v, w) \geq w(u, w)$.



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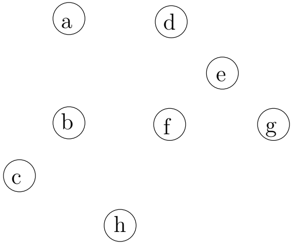

Status

The problem still remains NP-complete

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Time Complexity

Computing the minimum spanning tree using Prim's Algorithm needs $O(n^2)$ time (since G is a complete graph). All other works can be done in $O(n)$ time.

Analysis of Approximation Ratio

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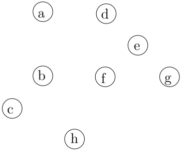
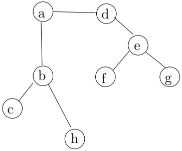
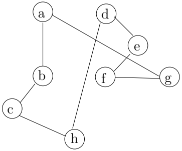
$H^* \rightarrow$ Optimal Tour. $H \rightarrow$ Tour produced by our algorithm.

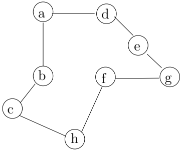




MST



Tour H



Optimal Tour H^*

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Thus $w(\mathbf{A}) = 2 \times w(T) \leq 2 \times w(H^*)$.
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We get a tour H from the walk A as follows:

For each vertex, we remove its second occurance in the walk except r.
By triangles inequality, we have $w(H) \leq w(\mathbf{A})$.
Thus we have $w(H) \leq 2 \times w(H^*)$. $\square$

Best Known Result on Δ-TSP

A 1.5-approximation algorithm for the Δ-TSP using maximum matching (Christofides 1976).

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Theorem

If $P \neq NP$ and $\rho \geq 1$, there is no polynomial time apprimation algorithm for the general TSP with ratio bound ρ.

Proof [By contradiction]

Suppose there is a ρ-approximation algorithm $\mathcal{A}$ for the general TSP problem. Let us assume that ρ is an integer.
We show that $\mathcal{A}$ can solve the hamiltonian cycle problem for any arbitrary graph $G = (V, E)$ in polynomial time.
Let $G' = (V, E')$ be a complete graph, with $E' = \{(u, v) | u, v \in V$ and $u \neq v\}$

$$w(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E, \text{ and} \\ \rho \times |V| + 1 & \text{otherwise.} \end{cases}$$

Assign $w(u, v) = 1$ if $(u, v) \in E$, and $\rho \times |V| + 1$ otherwise.

If the original graph has a hamiltonian cycle, the optimal tour will be of cost $|V|$.

Any non-optimal tour will be of cost at least $(\rho|V| + 1) + (|V| - 1) \geq \rho|V|$.

Proof (contd.)

Now we execute the algorithm $\mathcal{A}$ on the graph G'.

If G has a Hamiltonian cycle, then $\mathcal{A}$ must produce a tour in G' of cost at most $\rho|V|$.

But it is impossible unless it returns a tour in G' corresponding to the actual Hamiltonian cycle in G.

Thus we have a polynomial time algorithm for the Hamiltonian cycle problem.

Rectangle Stabbing Problem[1]

Problem Statement:

Given a set $\mathcal{R}$ of n axis-parallel rectangles, find the minimum number of axis-parallel lines to stab all the members in $\mathcal{R}$.

A rectangle $r \in \mathcal{R}$ is given using a pair of coordinates $[(a, b), (c, d)]$ corresponding to its (bottom-left, top-right) diagonal.

For the sake of simplicity, we assume that the coordinates are integer valued.

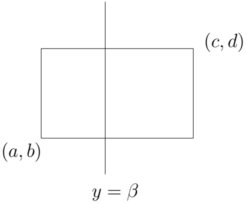
[1]Gaur, Ibaraki and Krishnamurthy, J. of Algorithms, 2002

Rectangle Stabbing Problem

Definition

An axis-parallel (horizontal/vertical) line ℓ stabs a rectangle $\mathbf{r} = [(a, b), (c, d)] \in \mathcal{R}$ if ℓ passes through the interior of the rectangle r. if

Example:



Here the line $y = \beta$ stabs the rectangle. This implies $a + 1 \leq \beta \leq c - 1$

Status of the Problem and Our Objective

Status:

The problem is NP-hard

Reference: Hasin and Megiddo, Discrete Appl. Math., 1991

Objective: To design a constant factor approximation algorithm

Result available:

A 2-factor approximation algorithm that runs in $O(n^5)$ time.

Tools used: LP-relaxation.

Integer Programming Formulation

Solution Space: $H \bigcup V$.

H – Set of $2n$ horizontal lines at $y = b_i + \epsilon$ and $y = d_i - \epsilon$, where b_i and d_i are y-coordinates of bottom and top boundaries of i-th rectangle.

V – Set of $2n$ vertical lines at $x = a_i + \epsilon$ and $y = c_i - \epsilon$, where a_i and c_i are x-coordinates of left and right boundaries of i-th rectangle.

Take 4n decision variables, namely $x_1, x_2, \ldots, x_{2n}$, $y_1, y_2, \ldots, y_{2n}$.

x_i – corresponds to i-th vertical line, and

y_j – corresponds to j-th horizontal line.

These decision variables can assume values 0 and 1 only.

Integer Programming Formulation

H_k – Set of horizontal lines that stab the rectangle r_k, and

V_k – Set of vertical lines that stab the rectangle r_k.

Integer Programming Problem – P

Objective Function:

min $\sum_{i \in V} x_i + \sum_{j \in H} y_j$

Constraints:

For each rectangle r_k, $k = 1, 2, \ldots, n$, we have the constraint

$\sum_{i \in V_k} x_i + \sum_{j \in H_k} y_j \geq 1$

$x_i \in [0, 1]$, for all $i = 1, 2, \ldots, 2n$, and

$y_j \in [0, 1]$, for all $j = 1, 2, \ldots, 2n$.

LP Relaxation

Linear Programming Problem – $\bar{P}$

Objective Function:

min $\sum_{i \in V} x_i + \sum_{j \in H} y_j$

Constraints:

$\sum_{i \in V_k} x_i + \sum_{j \in H_k} y_j \geq 1,$ for all $k = 1, 2, \ldots, n$

$x_i \geq 0,$ for all $i = 1, 2, \ldots, 2n$, and

$y_j \geq 0,$ for all $j = 1, 2, \ldots, 2n$.

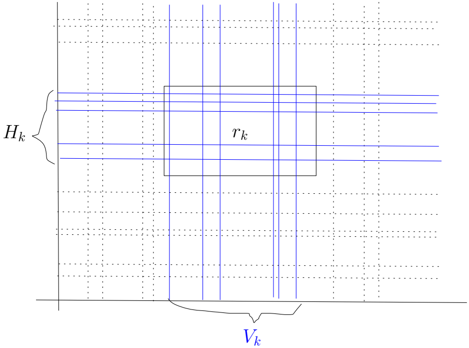
Analysis of LP Solution

Let $\bar{x}_i, i = 1, 2, \ldots, 2n$ and $\bar{y}_j, j = 1, 2, \ldots, 2n$
be an optimal fractional solution of the LP problem $\bar{P}$.

For each rectangle r_k, $k = 1, 2, \ldots, n$, we have

either $\sum_{i \in V_k} \bar{x}_i \geq \frac{1}{2}$

or $\sum_{j \in H_k} \bar{y}_j \geq \frac{1}{2}$.



Analysis of LP Solution

Let R_H be the set of all k such that $\sum_{i \in V_k} \bar{x}_i \geq \frac{1}{2}$, and

R_V be the set of all k such that $\sum_{j \in H_k} \bar{y}_j \geq \frac{1}{2}$.

Implication:

- The set of rectangles in R_H will be stabbed by horizontal lines, and
- The set of rectangles in R_V will be stabbed by vertical lines.

Thus, we have the following two problems

P_H: Compute the minimum clique cover of an interval graph with the vertical intervals corresponding to the rectangles in R_H, and

P_V: Compute the minimum clique cover of an interval graph with the horizontal intervals corresponding to the rectangles in R_V.

P_H and P_V can be optimally solved in polynomial time.

Analysis of Approximation Factor

Let

Q: Optimum solution of the integer programming problem P,

$\hat{Q}$: Optimum solution of the linear programming problem $\hat{P}$,

Q_H: Optimum solution of the clique cover problem P_H,

Q_V: Optimum solution of the clique cover problem P_V.

Theorem

$Q_H + Q_V \leq 2Q$.

Analysis of Approximation Factor

Proof: Let $\hat{Q} = (\hat{x}, \hat{y})$ be an optimal fractional solution of $\hat{P}$

$\implies Q_V^* = 2\hat{x}$ and $Q_H^* = 2\hat{y}$ are feasible solutions of P_H and P_V.

Reason: For every $k \in R_H$, we have $\sum_{i \in H_k} \hat{y}_i \geq \frac{1}{2}$
$\implies \sum_{i \in H_k} 2\hat{y}_i \geq 1$.

We have,

- $Q_H + Q_V \leq Q_H^* + Q_V^*$
  [since Q_H and Q_V are optimum solutons and Q_H^* and Q_V^* are feasible solutions of the same minimization problems.]
- $Q_H^* + Q_V^* = 2(\hat{x} + \hat{y}) = 2\hat{Q}$.
- $\hat{Q} \leq Q$
  [Since optimum solution of an LP minimization problem is less the optimum solution of its corresponding IP problem]

Thus, we have the proof of the result.