1. Give context-free grammars generating the following sets.

(a) The set of all strings of balanced parentheses, i.e., each left parenthesis has a matching right parenthesis and pairs of matching parentheses are properly nested.

(b) The set of all strings over alphabet \{a, b\} with exactly twice as many a’s as b’s.

(c) The set of all strings over alphabet \{a, b, \cdot, +, *\} that are well-formed regular expression over alphabet \{a, b\}. Note that we must distinguish between \(\epsilon\) as the empty string and as a symbol in the regular expression. We use \(\in\) in the latter case.

(d) The set of all strings over alphabet \{a, b\} not of the form \(ww\) for some string \(w\).

2. Suppose \(G\) is a CFG with \(m\) variables and no right side of production longer than \(l\). Show that if \(A \Rightarrow^* G \epsilon\), then there is a derivation of no more than \(l m - 1\) steps by which \(A\) derives \(\epsilon\). How close to this bound can you actually come?

3. Suppose \(G\) is a CFG and \(w\), of length \(l\), is in \(L(G)\). How long is a derivation of \(w\) in \(G\) if

(a) \(G\) is in CNF

(b) \(G\) is in GNF

4. Show that every CFL without \(\epsilon\) is generated by a CFG all of whose productions are of the form \(A \rightarrow a\), \(A \rightarrow aB\), and \(A \rightarrow aBC\).

5. A language \(L\) is said to have the prefix property if no word in \(L\) is a proper prefix of another word in \(L\). Show that if \(L\) is \(N(M)\) for DPDA \(M\), then \(L\) has the prefix property. Is the foregoing necessarily true if \(L\) is \(N(M)\) for a nondeterministic PDA \(M\)?

6. Show that the following are not context free languages.

(a) \(\{a^i b^j c^k | i < j < k\}\)

(b) \(\{a^i b^j | j = i^2\}\)

(c) \(\{a^i | i \text{ is a prime}\}\)

(d) the set of strings of a’s, b’s and c’s with an equal number of each

(e) \(\{a^n b^n c^n | n \leq m \leq 2n\}\)

7. Which of the following are CFL’s?

(a) \(\{a^i b^j | i \neq j \text{ and } i \neq 2j\}\)

(b) \((a+b)^* - \{(a^n b^n)^n | n \geq 1\}\)

(c) \(\{wwRw|w\text{ is in } (a+b)^*\}\)

(d) \(\{b_i \# b_{i+1} | b_i \text{ is i in binary, } i \geq 1\}\)

(e) \(\{wxw | x\text{ are in } (a+b)^*, w\text{ is in } (a+b)^+\}\)

(f) \((a+b)^* - \{(a^n b^n)^n | n \geq 1\}\)
8. Show that if $L$ is a CFL over a one-symbol alphabet, then $L$ is regular. [Hint: Let $n$ be the pumping lemma constant for $L$ and let $L \subseteq 0^*$. Show that for every word of length $n$ or more, say $0^n$, there are $p$ and $q$ no greater than $n$ such that $0^{p+iq}$ is in $L$ for all $i \geq 0$. Then show that $L$ consists of perhaps some words of length less than $n$ plus a finite number of linear sets, i.e., sets of the form $\{0^{p+iq} | i \geq 0\}$ for fixed $p$ and $q$, $q \leq n$. You may want to bound the number of parse trees of a certain depth, call them the base trees so that every other larger parse tree can be obtained by pumping some portions of one of the base trees.]