CSL 705 Theory of Computation, Tutorial Sheet 2

1. Use the recursion theorem to prove Rice's theorem on recursive index sets.

2. Show that a $T(n)$ time bounded $k$ tape Turing machine can be simulated by a 1 tape machine in $O(T^2(n))$ steps.

   Use this result to show how a Random Access Machine can be simulated by a 1 tape TM incurring at most a polynomial overhead.

3. An oblivious Turing machine is one such that the movement of the tape head at any step depend only on the length of the input and not the actual input. Show that every TM can be simulated by a 1-tape oblivious Turing Machine.

4. Consider the language $L = \{0^i \cdot 1^i, i \geq 0\}$. Can a 1-tape TM recognize this in $O(n)$ time?

5. If $L \in \mathcal{P}$, then which of the following are in $\mathcal{P}$

   (i) $\tilde{L}$

   (ii) $L^*$

6. Let $L_1, L_2 \in \text{PSPACE}$, then show that $L_1 \leq_{\text{PSPACE}} L_2$.

7. Show that any arbitrary boolean function of $k$ variables can be expressed by a CNF formula of at most $2^k$ clauses.

   Note: When $k$ is constant this is also constant, and therefore the CNF formula in the proof of Cook-Levin thm expressing the transition function of the NDTM is of bounded size as it only involves 4 cells.

8. Can you design an efficient algorithm that satisfies at least 50% of the clauses in a 3 CNF Boolean formula.

   How about 66%?

9. Consider the family of boolean circuits (consisting of AND, OR, NOT gates) of size $S$. Show that in order for the circuit to compute any of $n - ary$ boolean functions, $S$ should be $\Omega(2^n/\log n)$.

   Note that it shows that there are functions for for which exponential size circuits are necessary but these functions may not be in NP!

   Hint: Model the circuit as a labelled graph on $S$ nodes and use a counting argument on the possible number of circuits.

10. In the proof of Cook-Levin theorem, show that for $x \in L$, and the reduction function $f$, a certificate $y$ (non-deterministic choices) is mapped to a unique certificate of $f(x)$. Recall that $f(x)$ is a CNF formula and a certificate for CNF formula is a satisfying assignment.

11. Show that if a boolean formula in CNF contains at most one un-negated literal, then the satisfiability problem can be solved in polynomial time.

   Note: Such clauses are called Horn clauses

12. Assuming that 3D matching is NP complete, show that the problem of partition is NP complete.

   Hint: Work out the details of the proof outlined in class.

13. In the partition problem, if the sum of the integers are bounded by a polynomial, show that it is possible to design a polynomial time algorithm.

14. Show that the problem of independent set is NP complete.

15. Show that the problem of sub-graph isomorphism problem is NP complete. In this problem, the input are two graphs $G_1$ and $G_2$, does $G_1$ contain a copy of $G_2$ as a subgraph.