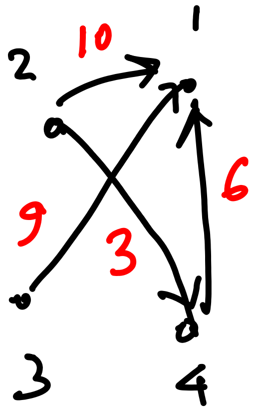
CSL 630 Lecture 3 July 31

$$C = A^{n \times n} \otimes B^{n \times n} \qquad c_{ij} = \min_k \{ a_{ik} + b_{kj} \}$$

A : adjacent matrix $a_{ij} = \omega(i,j)$



$$\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & \infty & \infty & \infty \\
2 & 10 & 0 & \infty & 3 \\
3 & 9 & \infty & 0 & \infty \\
4 & 6 & \infty & \infty & 0
\end{array}$$

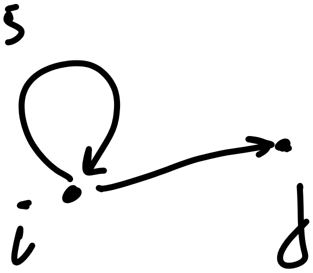
$A \otimes A \qquad a_{11} = 0$

$a_{12} = \infty \qquad a_{13}$

$A^2 =$

$$\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & \infty & \infty & \infty \\
2 & 9 & 0 & \infty & 3 \\
3 & 9 & \infty & 0 & \infty \\
4 & 6 & \infty & \infty & 0
\end{array}$$

$A^2_{i,j}$: the shortest path distance from vertex i to vertex j using atmost 2 edges



$\omega(i,i) + \omega(i,j)$

$\neq 0$

In case of self loops with weights

$c_{ij} := \min\{\omega_{ij}, \min_k\{\omega(i,k) + \omega_{k,j}\}\}$

A^3 : A^3_{ij} = the shortest path from i to j using atmost 3 edges

A^k, $k \le n-1$: A^k_{ij} : shortest path using atmost k edges

<u>Proof by induction on k</u> : h.w.-ex.

A^{n-1} : Computing x^n for some given no. x

$(5.8)^{20}$ $\quad \underbrace{5.8 \times 5.8 \cdots 5.8}$

$$x^n = x \to x^2 \to (x^2)^2 \to (x^2)^2$$

$$\cdot \to \quad x^{2^i}$$

$$\text{power}(x, n) = \begin{cases} \left(x^{n/2}\right)^2 & \text{if } n \text{ is even} \\ x \cdot \left(x^{\frac{n-1}{2}}\right)^2 & \text{otherwise} \\ 1 & \text{if } n = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } n = 0 \\ \text{power}\left(x, \frac{n}{2}\right)^2 & n \text{ is even} \\ x * \left(\text{power}\left(x, \frac{n-1}{2}\right)\right)^2 & n\text{: odd} \end{cases}$$

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(1)$$

$$T(1) = O(1) \qquad T(n) = O(\log n)$$

operations

Matrix powering of $\left(A^{n \times n}\right)^k$

$$T(k) = T\left(\frac{k}{2}\right) + O(n^3)$$

$$T(k) = O(n^3 \cdot \log k)$$

Floyd-Warshall algorithm stated in an algebraic framework

$O(n^3 \log n)$ steps

APSP matrix contains all pairwise S.P. distances

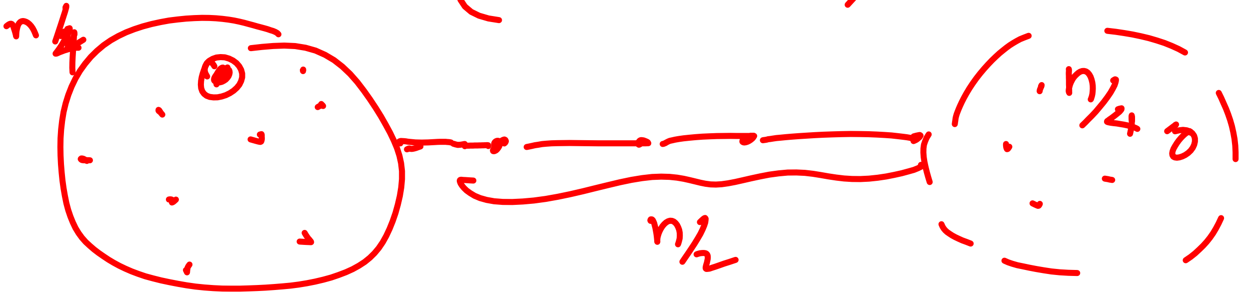
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & \cdots & j & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{matrix} & \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & \boxed{d_{ij}} & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

P_{ij} in the graph

$v_i \; v_{i+1} \cdots v_j$

$\omega(i, i_1)$

Storage: $O(n^2 \times n) = O(n^3)$



$$v_i \to v_{i_1} \to v_{i_2} \to v_{i_3} \cdots \to v_j$$

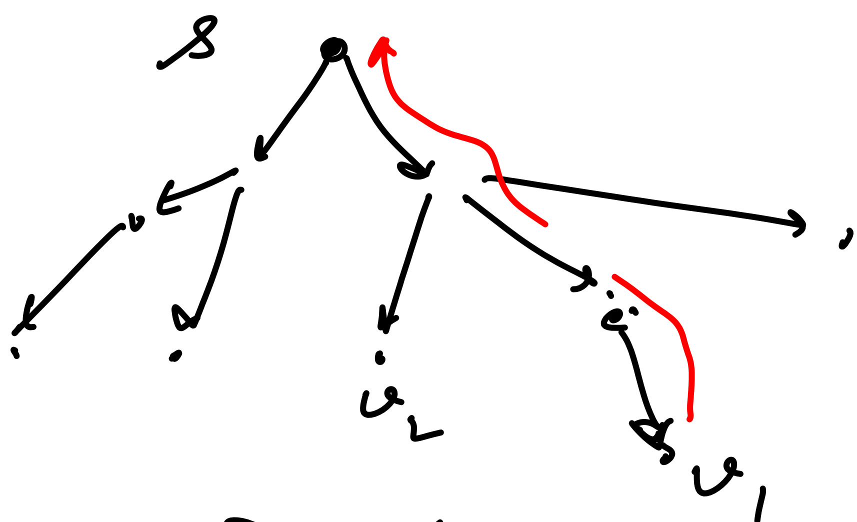
shorter

SSSP : Single source

What is the structure of the paths?

It is a "spanning" tree

Shortest path tree



Find a proof

Dijkstra's algorithm:

Heap: ① extract the key with min label

② After extracting we relax all edges incident to it.

(Decrease key) → Delete, Insert

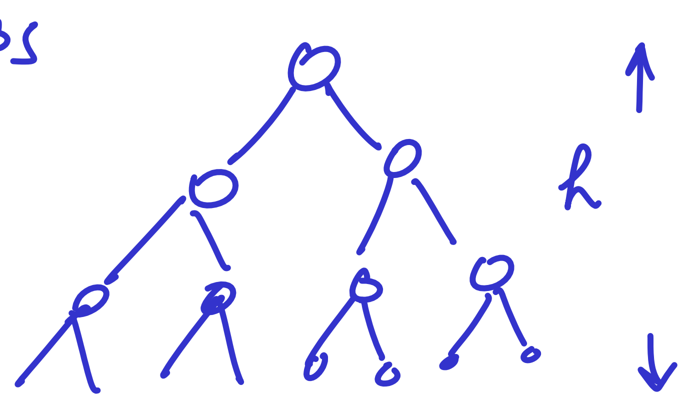
$$O\left(|V|\log|V| + |E|\cdot(\text{time decrease key})\right)$$

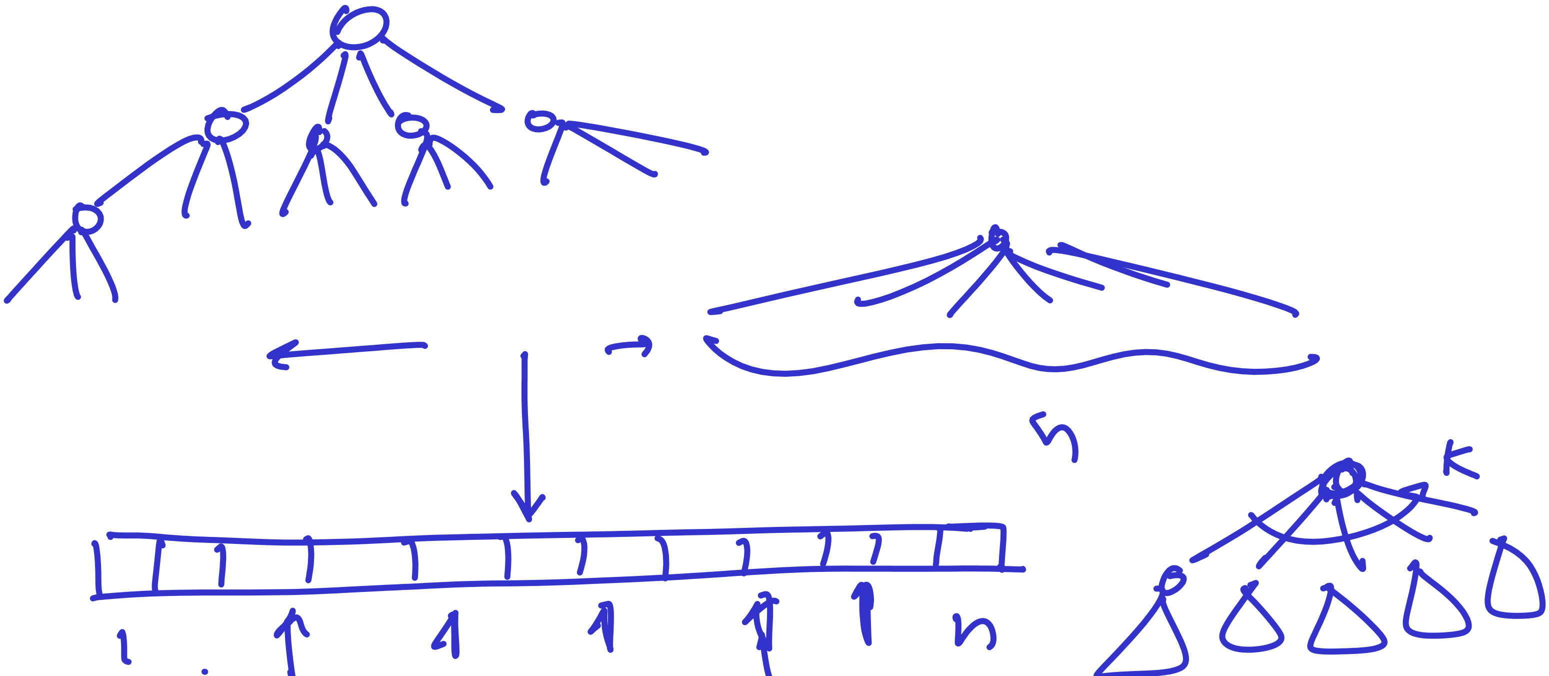
$\log|V|$

$$O\left((|E|+|V|)\log|V|\right)$$

Binary Heaps

(Min heaps)





K splitters

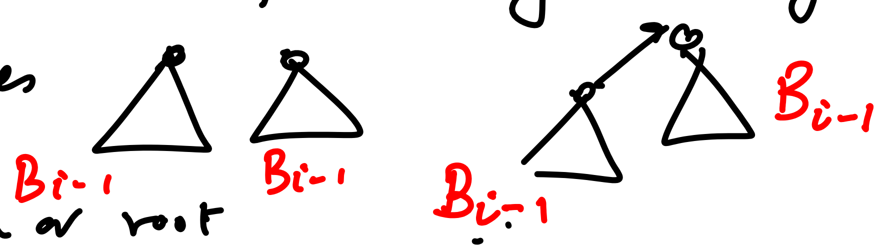
K-ary search

Family of Trees

Order i tree B_i

B_0 ● single node

B_i, $i > 0$ can be formed by taking two B_{i-1} trees



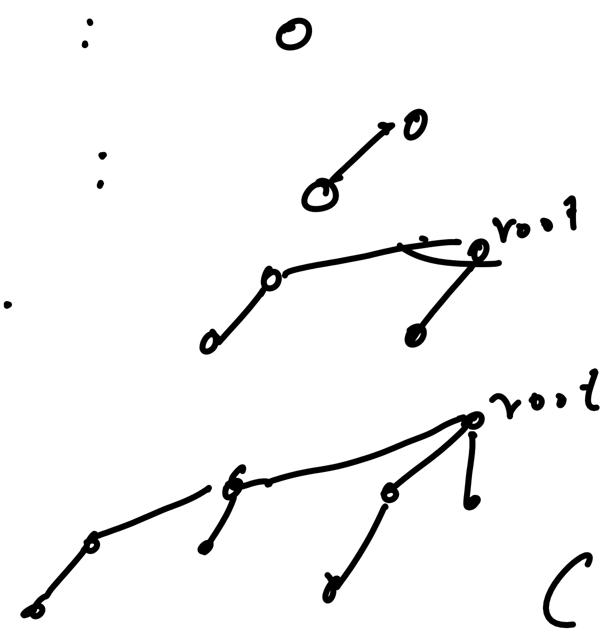
root becomes child of root

B_0 :

B_1 :

B_2 :

B_3 :



(Binomial)

Properties of such a family of trees

(i) B_i has 2^i nodes (verify by induction)

(ii) The root of B_i has i children

(iii) The no. of nodes in level k (from the root) of B_i has $\binom{i}{k}$ nodes