CSL 630 Lecture 29 Nov 17

The 3SAT problem is NPC

Given a 3CNF formula with m clauses over n boolean variables whether or not there exists a satisfiable assignment.

Variables $x_1 \; x_2 \; \ldots \; x_n$

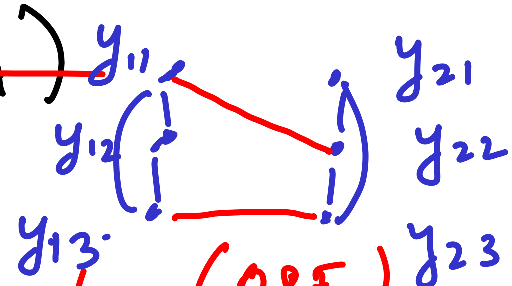
Literals $x_1, \bar{x}_1, x_2, \bar{x}_2 \ldots x_n, \bar{x}_n$

$$(y_{11} \vee y_{12} \vee y_{13}) \wedge (y_{21} \vee y_{22} \vee y_{23}) \ldots (y_{m_1} \vee y_{m_2} \ldots)$$

$$y_{i,j} \in \{x_1, \bar{x}_1 \ldots x_n, \bar{x}_n\}$$

Eg $(x_2 \vee \bar{x}_3 \vee x_5) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_5) \wedge$

$\cancel{(\bar{x}_3 \vee x_4 \vee \bar{x}_4)}$



$x_1 = T \quad x_2 = T$

Quantified Boolean Formula (QBF)

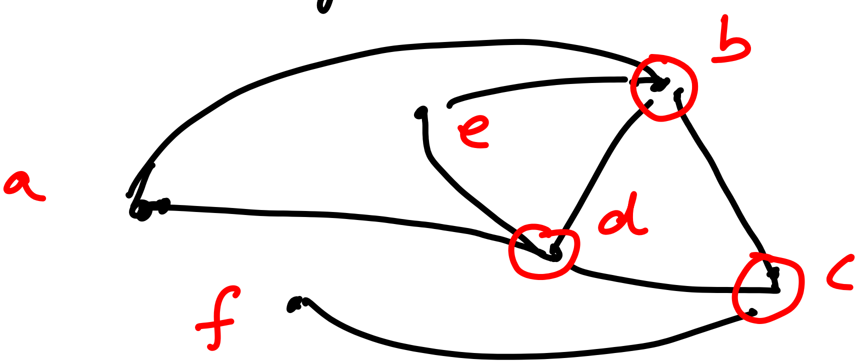
$\exists x_1 \, \forall x_2 \, \forall x_3 \, \exists x_4 \, \exists x_5$ (Boolean formula)

PSPACE-complete (polynomial space)

Vertex cover problem in graphs

Given any undirected graph $G = (V, E)$ and an integer K, $K \leq |V|$

Does there exist a subset $S \subset V$ s.t. $|S| = K$ and for all $w \in V - S$, there is atleast one edge $(w, u) \in E$ $u \in S$



$K = 3$ $\quad K = 2$?

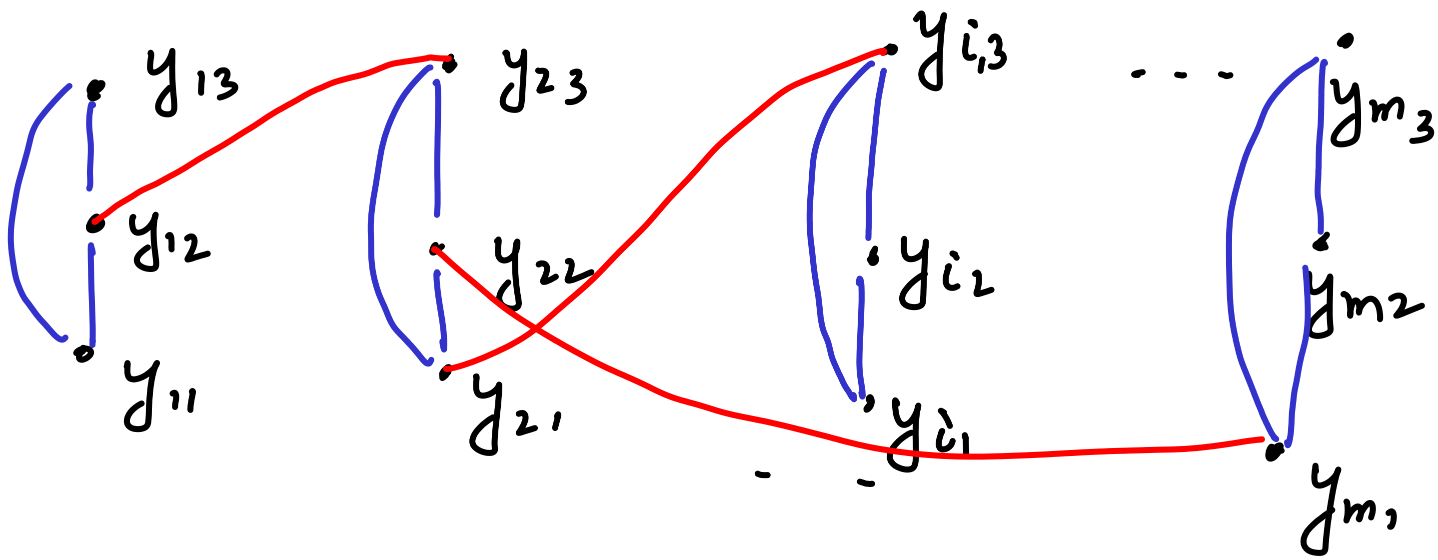
V.C. is in NP

We intend to prove NP completeness by demonstrating $3SAT \propto_{poly} V.C.$

Given any arbitrary 3CNF boolean formula I_1, we must construct an instance of V.C., say I_2 s.t. I_2 is true iff I_1 is satisfiable

$$f(I_1) = I_2$$

Given a 3CNF formula with literals $y_{11}, y_{12} \ldots$ we will define the following graph



There is an edge between $y_{i,s}$ and $y_{j,t}$ iff the corresponding literals are complements of each other.

$K = 2m$ recall $m = \#$ clauses

The reduction can be done in polynomial -time.

<u>Claim</u> : $f(I_1)$ has a vertex cover of size K iff I_1 is satisfiable

$\Rightarrow$ If I_1 is satisfiable then we have a vertex cover of size k

Then there exists a truth assignment, s.t., every clause has at least one literal set to true

Choose an arbitrary true literal in a clause (break ties arbitrarily) and select the remaining 2 "vertices" (corresponding to literals) in a set W

Claim : W is a vertex cover of size $2K$

- We cover all Δ edges
- At least one of the endpoints of a "cross-edge" must correspond to a False literal and hence must be in W

$\Leftarrow$ If $f(I_1)$ has a v.c. of size K then I_1 is satisfiable

$\Leftrightarrow$ if I_1 is not satisfiable $f(I_2)$ doesn't have a VC of size K

$A \Rightarrow B$

$\Leftrightarrow \bar{B} \Rightarrow \bar{A}$

<u>Obs</u>: Every Δ has exactly two vertices in the cover

We set the remaining literal to be True

Question: Is there a consistent truth assignment?

We cannot set both endpoints of cross edge to be true, otherwise we don't have a legal vertex cover.

Could we have left out any variables in the truth assignment

— NP completeness Guide —

Garey & Johnson

<u>Set cover problem</u>

Given a family of subsets $S_1, S_2, \ldots S_m$ over a ground set S of n elements are there k subsets in the family whose union is S?

<u>Claim</u> V.C. is a special case of a set cover problem

$\Rightarrow$ Set cover is also NPC

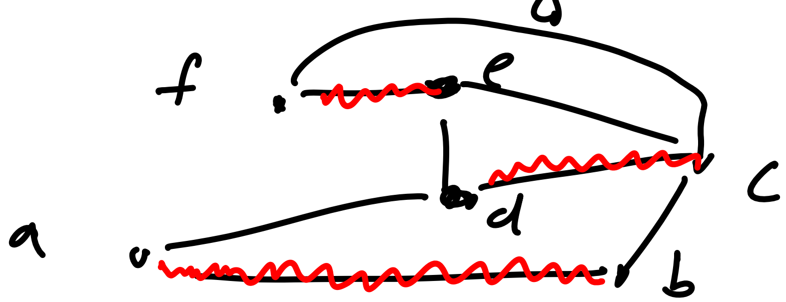
Even if we can't solve the exact optimization problem, can we design heuristics to give some "provable guarantees" on their performance

Exercise: How well does the greedy perform for Vertex Cover?

An alternate soln

1. We construct a "maximal matching"

We keep choosing edges such that no vertex has more than degree 1.



Claim: 1. $\{a, b, c, d, e, f\}$ is a vertex cover

2. It is no more than 2 times the minimum V.C.