CSL 630 Lecture 28 Nov 15

An NP complete problem π is
(i) in the class NP

NP hard (ii) All problems in NP are polynomial time reducible to π

$$\forall \pi' \in NP, \quad \pi' \propto_{poly} \pi$$

<u>Cook-Levin thm</u>: The satisfiability problem of Boolean formulae is NP complete.

Given a new problem π_1, try to establish (i). Suppose we can show that some NPC problem π_2 is polytime reducible to π_1

$$\underset{(SAT)}{\pi_2} \propto_{poly} \pi_1 \qquad \pi_1 \propto_{poly} \pi_2$$

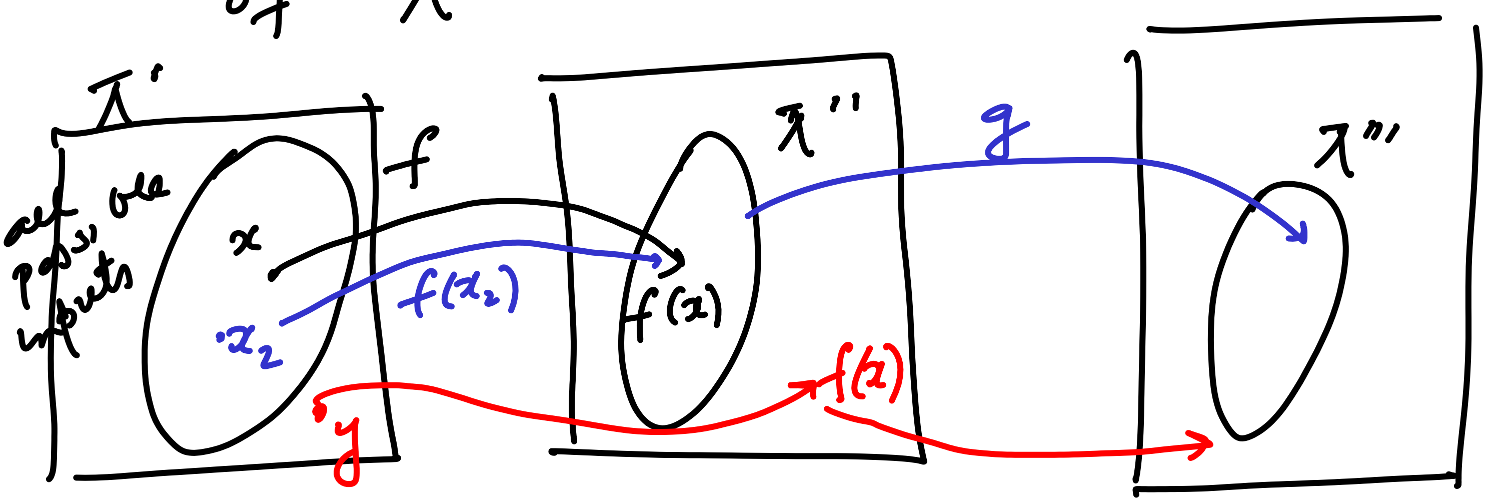
$$\left.\begin{array}{l} \pi_2 \propto_{poly} \pi_1 \\ \forall \pi \in NP \quad \pi \propto_{poly} \pi_2, \end{array}\right\} \pi \propto_{poly} \pi_1$$

Claim polytime reduction is transitive

If $\pi' \propto_{poly} \pi''$ and $\pi'' \propto_{poly} \pi'''$

then $\pi' \propto_{poly} \pi'''$

Consider any instance of π', say x
There exists a poly-time reduction function f s.t. $f(x)$ is an instance of π''



(i) x is a YES instance of π' iff $f(x)$ is a YES instance of π''

(ii) f is computable in polynomial time

$f: \Sigma^* \to \Sigma^*$

Σ : alphabet

Σ^* is the set of all strings

Claim $g(f)$ is a polytime redn function from π' to π'''

Composition of polynomials is a polynomial

$f(x) = x^3 \qquad g(x) = x^2$

$g(f(x)) = x^6$

length of the string

$|x|$. length of input x

Time to compute f is $|x|^3$

$f(x)$ is now of length $|x|^3$

g is applied to $f(x)$

Cook's thm was published in 1971

Karp published proofs of reduction for 6 very basic problems

(i) Hamilton Circuit

(ii) Coloring
3- colorable

(iii) Partition (iv) 3 dim matching

<u>3-CNF</u>

SATISFIABILITY OF BOOLEAN FORMULA IN CONJUNCTIVE NORMAL FORM with exactly 3 literals per clause is NPC

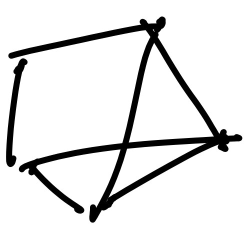
$(x_{11} \vee x_{12} \vee x_{13}) \wedge (x_{21} \vee x_{22} \vee x_{23}) \ldots$

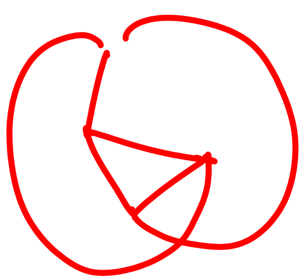
$(\qquad\qquad)$

literals meaning a variable x or $\bar{x}$

Are there problems in NP not known to be in P and neither known to be NPC ?

(1) Graph Isomorphism





Ladner's thm

If $P \neq NP$ then there are problems in NP which are neither NP complete nor in P

(2) Factorization

(3) ~~Primality~~ testing

Partition problem

Given set S of n integers $x_1, x_2 \ldots x_n$
can we partition S into S_1, S_2 s.t.

$$\sum_{i \in S_1} x_i = \sum_{i \in S_2} x_i$$

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3, 8, 20, 5, 4, 9

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Partition problem is known to be NPC

Knapsack problem

Given profits $p_1, p_2, \ldots p_n$
weights $w_1, w_2 \ldots w_n$
Knapsack capacity B
maximize profit

Partition $\propto_{poly}$ knapsack

(If we succeed then Knapsack is NP-H but not NPC)

Given any instance I of the partition problem we can map it to an instance I' of knapsack s.t. I' can provide the soln to I

$$S \xrightarrow{f} \begin{array}{l} p_1, p_2 \cdot \quad p_n \\ w_1, w_2 \cdots \quad w_n \\ B \end{array}$$

n integers

$x_1, x_2 \cdots \quad x_n$

$$B = \frac{\sum x_i}{2} \qquad\qquad \max \sum p_i y_i$$

$$w_i = x_i \qquad p_i = \cancel{1}\ x_i \qquad y_i \in \{0,1\}$$

<u>Claim</u> : The knapsack has profit B
iff S has a partition.

The reduction can be done in polynomial
time