CSE 630 Lecture 12 Sept 8

The following are equivalent

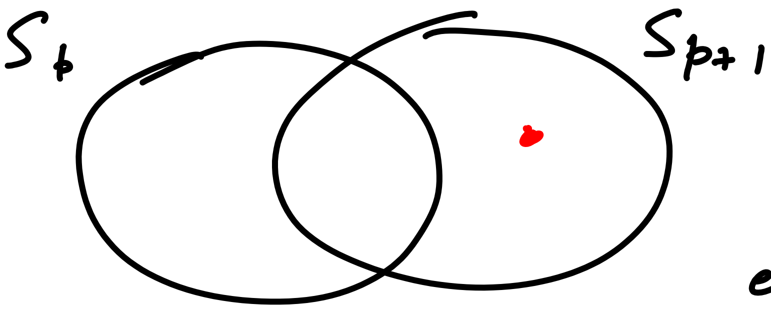
1. $M = (S, \mathcal{I})$ is a matroid
   (i.e. greedy works)

2. For $S_p, S_{p+1} \in \mathcal{I}$ with p and
(exchange property) $p+1$ elements resp., there exists
   $e \in S_{p+1} - S_p$ s.t. $S_p \cup \{e\}$ is independent

3. For $A \subseteq S$, all <u>maximal</u> independent
   subsets of A have the same
   cardinality

① ⇒ ② ⇒ ③ ⇒ ①

Proof by contradiction, greedy works but for some subsets S_p, S_{p+1} we cannot augment S_p with an element from S_{p+1}



Let us assign weights to the elements of $S_p \cup S_{p+1}$ as follows

$$w(e) = \begin{cases} p+1 & \text{if } e \in S_p \\ p & \text{if } e \in S_{p+1} - S_p \\ 0 & \text{otherwise} \end{cases}$$

Greedy chooses all elements of S_p
can't choose from $S_{p+1} - S_p$ (not ind.)
and rest of the elements have wt 0, so
soln has wt $(p+1)p$

By choosing all elements of S_{p+1}, we
get $\geq \underbrace{p}_{\text{at least one extra}} + p \cdot (p+1)$

Contradiction of greedy

2 ⇒ 3

Proof by contradiction

Suppose two maximal subsets P, Q have different card. $|P| < |Q|$

Then by exchange property we should be able to augment P from Q, i.e. P is not maximal

3 ⇒ 1

All maximal subsets have the same cardinality ⇒ greedy must work

Let us arrange the elements in decreasing order of wts $w(e_i) \geq w(e_{i+1})$

$$
\begin{array}{lccccccc}
\text{Opt} & e_1 & e_2 & e_3 & e_{i-1} & \boxed{e_i} & & e_{n_1} \\
\text{Gre} & \overset{\shortparallel}{e_1'} & \overset{\shortparallel}{e_2'} & \overset{\shortparallel}{e_3'} & e_{i-1}' & \overset{\vee}{e_j} & & e_{n_2}
\end{array}
$$

$wt(e_j) < w(e_i)$ ⟸ $n_1 = n_2$ because of 3

because greedy soln is < optimal soln

$A = \{ e \mid w(e) \geq w(e_i) \}$

maximal subset in A

$\{e_1', e_2', \ldots e_{i-1}'\}$ must be maximal otherwise greedy would have picked up more elements from A before e_i'. But that is a contradiction since the maximal ind set in A is at least of size i.

Maximum Spanning Tree

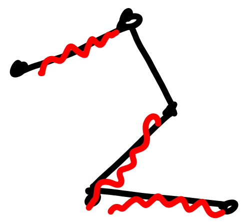
Let us show that ② is satisfied i.e. We have two forests F_p and F_{p+1} with p and $p+1$ edges we must show that there is some edge $e \in F_{p+1} - F_p$ s.t. $F_p \cup \{e\}$ is a forest

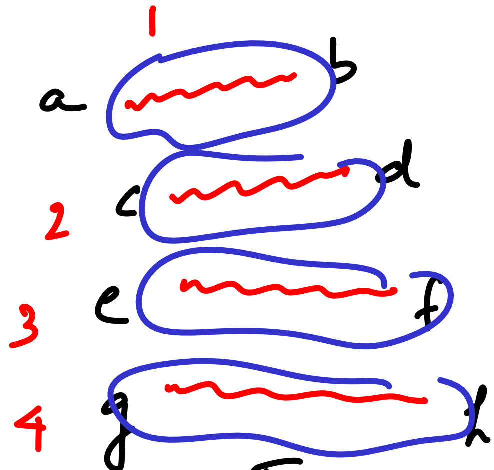
Case 1 $\quad V(F_{p+1}) > V(F_p)$

We can add any edge incident on the extra vertex in $V(F_{p+1})$ to F_p safely.

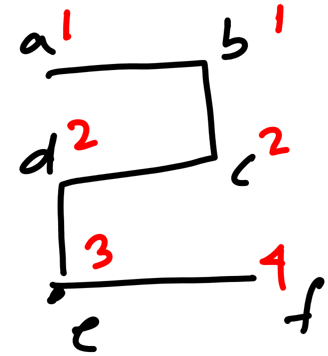
Case 2 : $V(F_{p+1}) \leq V(F_p)$

F_{p+1}





F_4



F_5

$$V(F_{p+1}) \subseteq V(F_p)$$

We want to add an edge from F_{p+1} to F_p that spans different components

Obs 1 : The # components in F_{p+1} must be $<$ # components in F_p

In F_{p+1} mark the component no. of F_p with each vertex. Same component in F_{p+1} must have vertices from different components of F_p.
This implies that there must be an edge in F_{p+1} connecting two comp of F_p.