We have a ground set $S$ of $n$ elements. Let $M$ be a family of subsets, $I \subseteq 2^S$ (power set). $I_j$ are those subsets that are "feasible." We have a weight function $W : S \rightarrow \mathbb{R}^+$.

For a subset $P \subseteq S$

$$W(P) = \sum_{x \in P} W(x)$$

Objective: Find the largest weight subset from $I$.

All (maximization) optimization problems can be formulated in this framework.

For example: Knapsack problem

$S: \{x_1, x_2, \ldots, x_n\}$
\[ I : \text{all subsets of objects such that their weight is } \leq B \]

\[ \sum_{x_i \in P} v_i x_i \leq B \]

Maximize: \[ \sum w(x_i) \]

\[ w(x_i) = w_i \]

Minimum Spanning Tree
(Maximum Spanning Forest) \( G = (V, E) \)

\[ S : \text{ set of edge } E \]

\[ I : \text{all subset, } I \text{ edges that do not have an induced cycle, i.e. all forests} \]

Objective function: Find the maxon weight forest.

Observation: If the graph \( G = (V, E) \) is connected then the solution is a tree (connected)
Eg. Matching in Graphs (Bipartite)

Find a subset of edges such that no vertex has two edges incident.

Maximum Cardinality Matching (MCM)
Find a matching with maxm edges.

Maximum Weight Matching (MWM)
Each edge has a weight and we pick the matching with maxm weight.

S: \( E \) & J: edges without common endpoints

Obj function for MWM: pick the maxm cut subset from J

Half matching:
Pick a subset of edges so that no vertex has more than one incoming edge.
Generic greedy algorithm

\[ S: \ \text{grant set of elements } e_1, e_2, e_3 \ldots e_m \]
\[ J: \ \text{family of "independent" subsets} \]
\[ W: \ \text{weight function } W: S \rightarrow \mathbb{R}^+ \]

Objective: find a subset \( P \subset S \), \( P \in J \)
\[ W(P) \text{ is maxm.} \]

Initialize \( T = \emptyset \)

Order the elements in decreasing sequence of weights (assume \( w_1, w_2, \ldots \) is ordered)

for \( i = 1 \) to \( m \) do

\[ T(i) \left[ \begin{array}{l}
\text{if } T \cup \{e_i\} \in J \text{ then} \\
T \leftarrow T \cup \{e_i\}
\end{array} \right] \]

Output \( T \)

When is \( T \) the optimal soln?

Greedy succeeds for a problem if it solves all instances.
Subset system \( M : (S, I) \)

M is a subset system if
\( \forall P \in I, \ Q \subseteq P \ Q \in I \)

M is a matroid if
generic greedy solves the optimization problem for any weight function.

The following are equivalent

1. \( M : (S, I) \) is a matroid

2. Exchange property: For all
\( P, Q \in I, \ |P| > |Q| \)
\( \exists \ e \in P - Q \text{ s.t. } Q \cup \{e\} \in I \)

3. Let \( A \subseteq S \)
   Then all maximal subsets of
   A have the same cardinality.