A problem $\Pi$ is $\text{NP}$ hard if all problems in the class $\text{NP}$ can be polynomially reduced to $\Pi$.

10. If $\Pi' \in \text{NP}$ then $\Pi' \leq_{\text{poly}} \Pi$

If $\Pi \in \text{NP}$ then $\Pi$ is $\text{NP}$ complete.

$\text{NP}$ hard/complete "under" polytime reduction.
Recall that polynomial reduction satisfy transitivity

\[ \pi' \leq_{\text{poly}} \pi'' \quad \text{and} \quad \pi'' \leq_{\text{poly}} \pi^* \]

\[ \implies \pi' \leq_{\text{poly}} \pi^* \]

If \( \pi \) is NPC and there is a polynomial algorithm for \( \pi \)

\( \rightarrow \) all problems in NP.

If we can show that there cannot exist a polynomial algorithm for \( \pi \), then \( P \neq NP \)

How does one show the \( \pi \) is NP complete? ??
Suppose $\Pi^1 \in \text{NPC}$ and $\Pi^2 \in \text{NPC}$

$\Rightarrow \quad \Pi^1 \leq_{\text{poly}} \Pi^2$

and $\Pi^2 \leq_{\text{poly}} \Pi^1$

To show a new problem $\Pi^3 \in \text{NPC}$

1. $\Pi^3$ is in $\text{NP}$

2. $\Pi^1 \leq_{\text{poly}} \Pi^3$

(since $\Pi \in \text{NP}$ $\Pi \leq_{\text{poly}} \Pi^1$

**Cook-Levin theorem**: The satisfiability problem of Boolean formula is $\text{NPC}$. 
Given $n$ boolean variables
say $x_1, x_2, \ldots, x_n$

$x_i \in \{T, F\}$

then, given any boolean formula

\[
\neg \land \left( x_1 \lor x_3 \right) \land \left( x_4 \lor x_5 \right)
\]

is there an assignment of $x_i$'s such that the expression is True.

Cook-Levinth[sh]er (stronger): The satisfiability problem of a boolean formula given as Conjunctive Normal Form (CNF) is NP complete with exactly 3 literals per clause.
\[(x_1 \lor \overline{x}_2 \lor x_3) \land (x_2 \lor \overline{x}_3 \lor x_4) \land (\quad) \quad \cdots \quad (\quad)\]

say \( m \) clauses each having 3 literals (a literal is a boolean variable or its complement)

\( 3 \) CNF formula
Vertex Cover problem

Does every edge have at least one of its endpoints in the cover (marked by red)?

Does there exist a vertex cover of size \( K \) (\( K \leq n \))? Is \( \text{v.c.} \) in \( \text{NP} \)?
It suffices to show that
$3CNF \leq_{poly} V.C.$

Given any instance $F$ of the $3CNF$ problem, say a formula $F$ we have to map it to some instance of the V.C. problem, say $P(F)$ such that $P(F) \rightarrow G$ such that $G$ has a vertex cover of size $k$ iff $F$ is satisfiable.

and $P$ must be computable in polynomial time.