Computational Geometry, Lectures 1,2
Art-Gallery Problem

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1 Art-Gallery Problem

The gallery will be represented by a simple polygon. There are no holes in the gallery, i.e. the interior of the gallery and its exterior are one connected components. The walls of the art gallery are made up of n straight line segments which are not necessarily orthogonal.

Guards are assumed to have a viewport of 360 degrees. Put otherwise they rotate at infinite speed. Moreover they can see as far as nothing is in their way, i.e until there is a wall. Guards will be represented by points in the region enclosed by the polygon. The guard sees all points it can be connected to by a segment lying entirely inside the polygon. Also, we assume that the guards are stationary.

We need to place guards such that every interior point should be visible to at least one guard. How many number of guards are needed for a gallery with n sides/vertices?

2 Art-Gallery Theorm (Chavatal)

If there are n edges/vertices, then $\lfloor n/3 \rfloor$ guards are necessary (for some) and sufficient (for all).

3 Proof of Art-Gallery Theorm

Here we prove the Art-Gallery Theorm. First of all, we prove some lemmas which will be helpful in proving the theorm.

3.1 Lemmas

3.1.1 Lemma-1

n guards suffice for whole gallery.

Proof: The proof is trivial. Simply place one guard at each vertex of the polygon.

3.1.2 Lemma-2

The vertices of a triangulation can be 3-coloured.

Proof:

First of all, we look at the dual of the triangulation. The dual of the triangulation is a graph G where each vertex represents a triangle of the triangulation and there is an edge between two vertices if those two corresponding triangles share a common edge. We claim that such a graph is a tree. Why is this a tree?

A graph is a tree if:

1. It contains no cycles.
2. It is connected : There is a path connecting any two vertices in the graph.

It is clear that all nodes are connected since all triangles are joined by diagonals. Thus from a node you can reach any other by passing over diagonals of adjacent triangles.

If there were cyles the polygon would have holes. These are not simple polygons since they have more they consist in more then one polygonal chain.

Thus, we have a tree. Each tree has at least two leaf nodes. Thus, our triangulation has at least two ears. Of course to be clear we must define what is a polygonal ear. We say that a polygon P has an ear at vertex V if the triangle formed by V and its two adjacent vertices lies inside P and does not contain a piece of the polygon. To be precise, there are no vertices of P inside the triangle formed by V and its 2 adjecent vertices.

Proof of 3-colouring theorm:

The proof is by induction on the number of vertices of the polygon P.

Base Case: (n=3)

Trivially, a triangle can be 3-coloured by assigning one colour to each vertex.

Induction Hypothesis:

Any triangulation with less than n vertices can be 3-coloured.

Induction Step:

Since any triangulation has two ears, pick any one of the ears. Let its vertex be V. Let its adjacent vertices be $V1$ and $V2$. The triangulation of the remaining $n-1$ vertices can be 3-coloured. Now, assign the colour to V different from the colour of $V1$ and $V2$. In this way, we have 3-colouring of the triangulation.

3.1.3 Lemma-3

Every polygon can be triangulated.

Proof: We claim that we can always find a diagonal in a polygon such that it does not intersect any boundary. If we are able to find the diagonal, the we can recursively keep on applying this argument on two polygons formed and hence triangulate the polygon.

Proof that we can always find a diagonal:

Pick one of the vertices of the polygon (say, lowest one a) and pick its adjacent vertices b and c.

1. If the line joining b and c does not intersect any boundary, then we are done. This is one such diagonal.

2. If the line joining b and c intersects a boundary, then draw a line parallel to the line joining b and c and shift it towards a. At the point when it leaves the boundary, it will leave at a vertex, d. Then the line joining a and d is such a diagonal which can't intersect any bounary. Hence, we have found the diagonal.

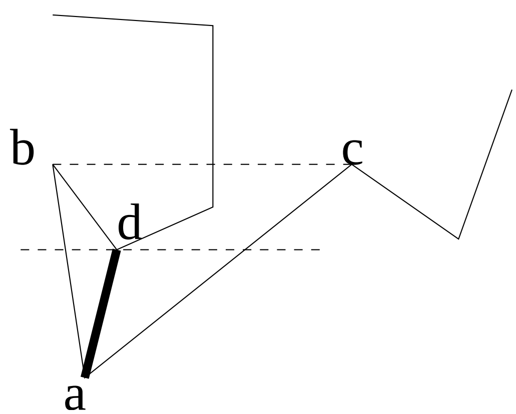

Figure 1: Lemma-3 Case2

3.2 Proof that $\lfloor n/3 \rfloor$ guards are sufficient for all

Triangulate the polygon and by the 3-colouring theorm, assign the coloures to the vertices of polygon. Let these colours be R,G and B.

num-R + num-G + num-B = n

Choose the colour which has got least number of occurances. Let it be R.

So, num-$R \leq n/3$

or, num-$R \leq \lfloor n/3 \rfloor$ since the number of its occurances must be an integer. This proves that $\lfloor n/3 \rfloor$ guards are sufficient for all.

3.3 Proof that $\lfloor n/3 \rfloor$ guards are necessary for some

For this look at a comb structure for an example. This has 12 sides and needs 12/3 or 4 guards.

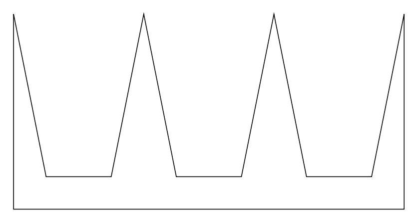

Figure 2: Comb Structure