Computatinal Geometry

Lecture 7,8
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1 Lower Bounds

We prove lower bounds for the problems studied so far.

1.1 Closest-Pair Problem

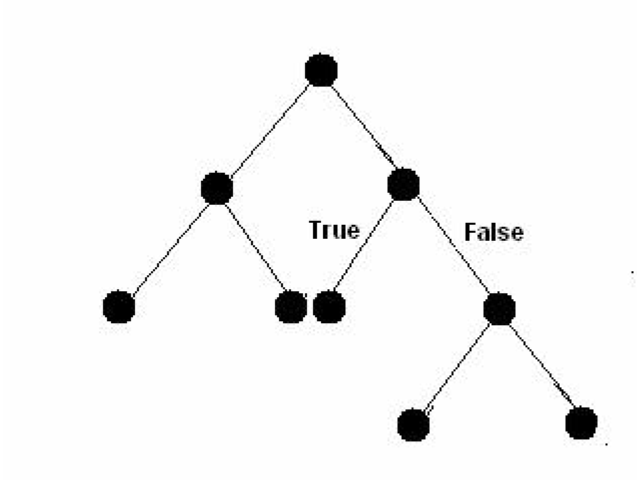
It is an $O(n \log n)$ problem. We show that we really need $\Omega(n \log n)$ operations.
For this we consider
Element Distinctness Problem.
Consider a set of n points $\{x_1, x_2, x_3, x_4, ..., x_n\}$. Are they distinct? We can map element distinctness problem to closest pair problem. Put $\{x_1, x_2, x_3, x_4, ..., x_n\}$ on a straight line. Calculate closest distance between the points. If closest distance = 0 then the elemnets are not distinct. If we show that **Element Distinctness Problem** $\geq$ T(n) then we can show that **Closest Pair Problem** $\geq$ T(n).

We need to specify the model of computation. A model of computation specifies
i)What are the operations that can be performed ?
ii)What operations can be counted as unit operation ?

2 Models of Computation

2.1 Comparison Tree Model

In comparison tree model we can only do comparison operations.The comparison operations are $\leq, \geq, , <, >, =$. We remove down the tree depending on True or False. At the leaf the answers are stored. For instance, For sorting No. of leaves $\geq n!$



But in geometrical problems we need more complex operations like square , square root etc. So we define Computation tree Model.

2.2 Computation Tree Model

In this model the following operations are considered to be unit operations
1.)Comparisons $\leq, \geq, <, >, =$
2.)a op b where op can be $+, -, /, *, ...$
3.)Memory indirection, array access.
4.)square root, $k^{th} roots, \log, \exp$
We do computation at some nodes and comparisons at the others as we go down the tree. Worst case computation time will occur at the maximum height.

3 Decision Problem

Consider $\mathbf{W} \subseteq \mathbb{R}^n$
Does $x \in W$? where x is a vector.

Element Distinctness
Let **W** = $\{x_1, x_2, ..., x_n : \prod_{i,j,i \neq j}(x_i - x_j) \neq 0 \}$
Let $x = (x_1, x_2, x_3, ..., x_n)$ be a set of points on a line , but we treat it as a single point in n-dimensional space.
The complexity of this problem depends on the number of connected components of **W**.

A subset **S** of $\mathbb{R}^n$ is said to be *connected* if for every pair of points x,y $\in$ **S**, there is a continuous arc joining x to y which lies completely inside **S**.
Connectivity is an Equivalence Relation.

We can partition **W** into connedted sets $\{\mathbf{W_1}, \mathbf{W_2}, \mathbf{W_3}, ..., \mathbf{W_k}\}$ such that any two points in $W_i, W_j, i \neq$

j are not connected. The difficulty of decision problem depends on the number of connected components of **W**.
For Instance,In 3-dimensional case n=3,there will be 6 connected components corressponding to the six possible orderings of $x_1, x_2, x_3, x_4, x_5, x_6$.

$$\begin{cases} W_1 = \{(x_1, x_2, x_3) : x_1 < x_2 < x_3\} \\ W_2 = \{(x_1, x_2, x_3) : x_1 < x_3 < x_2\} \\ . \\ . \\ . \\ W_6 \end{cases}$$

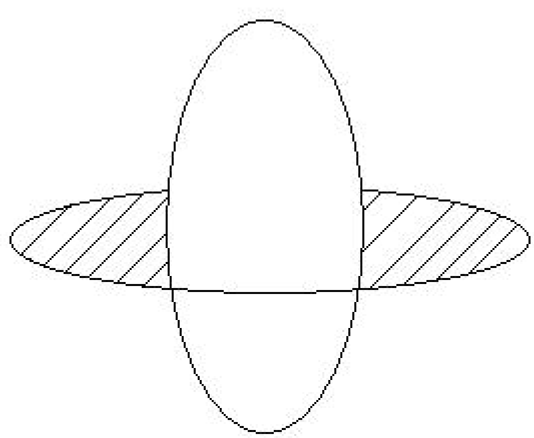
Claim: If we take a point in W_1 and another point in W_2, then they can't be connected by an arc lying in **W.**
So for n-dimensional case
Number of connected components $= n!$

Consider an Algebraic Computation Tree and the decision problem **W** $\subseteq \mathbb{R}^n$
Does $x \in W$? Complexity depends on the number of connected components of **W**

At every leaf node the corressponding sets of points are disjoint.If the constrains are non-linear then the set represented by a leaf node may not be a connected component.

Example

$$\begin{cases} x_2 - {x_1}^2 > 0 \\ {x_1}^2 + {x_2}^2 \leq 5 \end{cases}$$



Suppose all the functions are linear. Linear Computation Tree Model.
Constraints $f_v(x_1, x_2, x_3, ..., x_n)$ are of the form $a_1x_1 + a_2x_2 + ... + a_nx_n \geq 0$
Then the leaf nodes will be connected components.This is because these constraints are basically planes, so there intersections would will divide the region into convex sets. Therefore,
Number of Connected Components $\geq$ Number of connected components

$\Rightarrow NumberofLeaves \geq NumberofConnectedcomponentsof\mathbf{W}$
$\Rightarrow HeightoftheTree \geq \log_2(NumberofconnectedcomponentsofW)$

Consider Element Distinctness Problem
$W = \{(x_1, x_2, x_3, ...x_n)$:all coordinates are different$\}$
$W^c = \{(x_1, x_2, x_3, ...x_n)$:at least 2 coordinates are same$\}$
We see that the number of connected componenets in $W^c = 1$. Choose the problem that has a higher number of connected components.
In the constraints like $f_v(x_1, x_2, x_3, ..., x_n)$ causes a concern as when we branch on $' =' $ then the set of points that are not equal may not be convex. So we increase the height of the tree by 1. For the false branch on $' ='$ we further divide the set of points into two sets that are $' <'$ and $' >'$ or we can simply increase the degree of tree and on $=$ we can branch into three sets that are $=, >, <$. Converting a degree 3 tree to a degree 2 tree can at a maximum double the height of the tree.

$$\begin{cases} f_1(x_1, x_2, x_3, ..., x_n) = 0 \\ f_2(x_1, x_2, x_3, ..., x_n) \geq 0 \\ f_3(x_1, x_2, x_3, ..., x_n) > 0 \\ . \\ . \\ . \\ f_k(x_1, x_2, x_3, ..., x_n) < 0 \end{cases}$$

If these constraints f are non-linear then what is the number of connected components ? We have the following theorem

Milnor Theorem
Consider the following equlity functions in $\mathbb{R}^n$

$$\begin{cases} g_1(x_1, x_2, x_3, ..., x_n) = 0 \\ g_2(x_1, x_2, x_3, ..., x_n) = 0 \\ g_3(x_1, x_2, x_3, ..., x_n) = 0 \\ . \\ . \\ . \\ g_p(x_1, x_2, x_3, ..., x_n) = 0 \end{cases}$$

Number of connected components in W $\leq d.(2d-1)^{n-1}$ d: maximum degree of $g_1, g_2, g_3, ...g_p$

Above theorem is applied in Computational Geometry with slight variation

BanOr Theorem
Consider the following functions in $\mathbb{R}^n$

$$
\begin{cases}
q_1(x_1, x_2, x_3, ..., x_n) = 0 \\
. \\
. \\
. \\
q_r(x_1, x_2, x_3, ..., x_n) = 0 \\
p_1(x_1, x_2, x_3, ..., x_n) > 0 \\
. \\
. \\
p_s(x_1, x_2, x_3, ..., x_n) > 0 \\
p_{s+1}(x_1, x_2, x_3, ..., x_n) \geq 0 \\
. \\
. \\
p_h(x_1, x_2, x_3, ..., x_n) \geq 0
\end{cases}
$$

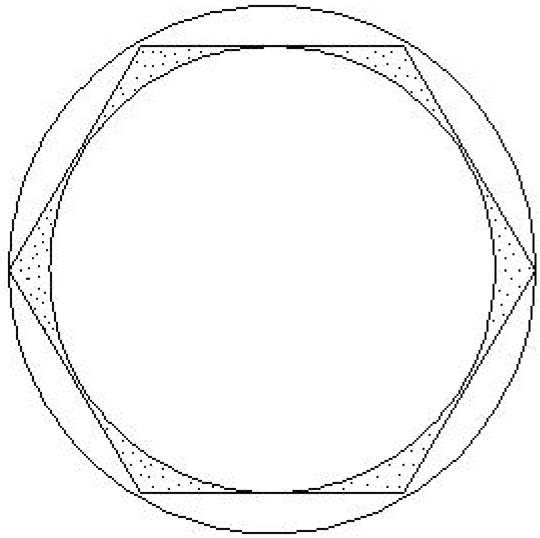
Number of connected components of **W** is bounded by $\leq d.(2d-1)^{n+h-1}$
In this case it depends on the number of inequalities.
Height of the tree of the tree is at most (r+h).
Each leaf can have several connected components $\leq d.(2d-1)^{n+h-1}$
Depth of the Leaf $\leq (r+h)$
If h^* is the height of the tree then
No. of leaves$\leq 2^{h^*}$
$\Rightarrow$No. of connected components $\leq 2^{h^*}.d(2d-1)^{n+h-1}$
since $h^* \geq h$
$\Rightarrow h = \Omega(\log(No.of connected componenets of W) - n)$

4 Application of Decision Problem to convex Hull

Problem 1: Given 'n' points in a plane, and a number h $\leq$ n, are the h points on the convex hull CH(S) ?
Answer : This construction problem is at least as hard as this decision problem. A lower bound on thid problem will also be a lower bound on the construction problem.

Lets look at a related problem

Problem 2: Consider a regular h-gon and a circle inscribed in the h-gon and another circle circumscribing the h-gon. Place the remaining n-h points arbitrarily in the wedges.



Do the n given points lie
1. Within the circumcircle
2. Outside the inscribed circle.
3. Have exactly h points on the convex hull.
In linear time we can reduce problem 2 to problem 1. Hence a lower bound on problem 2 implies a lower bound on problem 1.
Total number of configurations satisfying all the three conditions $= h^{n-h}$

Claim: Each configuration must be a separate connected component of the solution space.

Each path in higher dimensional space corressponds to a point going from one configuration to the other. We can't move along the edges of the convex hull because then it touches the incircle at a point (violating condition 2)