CSL852: Computational Geometry
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Dual Transforms

1 Introduction

This lecture is about solving the problems of constructing delaunay triangulation, finding the convex hulls, and constructing the first and the higher order voronoi diagrams with the help of dual transforms.

2 Delaunay Triangulation

The dual of voronoi diagram is obtained by denoting every face by a vertex and adding an edge between two vertices if and only if the corressponding faces in the voronoi diagram share an edge. The dual, thus, created has some useful properties. No four vertices in the delaunay triangulation lie on the same circle. This property corresponds to the assumption made during the construction of voronoi diagram that every voronoi corner must be of degree 3. Another useful property being that every circumcircle drawn on a delaunay triangle is empty and contains no delaunay vertex inside it. Using these properties, direct methods, similar to divide and conquor used in construction of voronoi diagrams, have been developed to construct delaunay triangulation without constructing the intermediate voronoi diagrams.

3 Lifting transform to construct the Voronoi diagrams

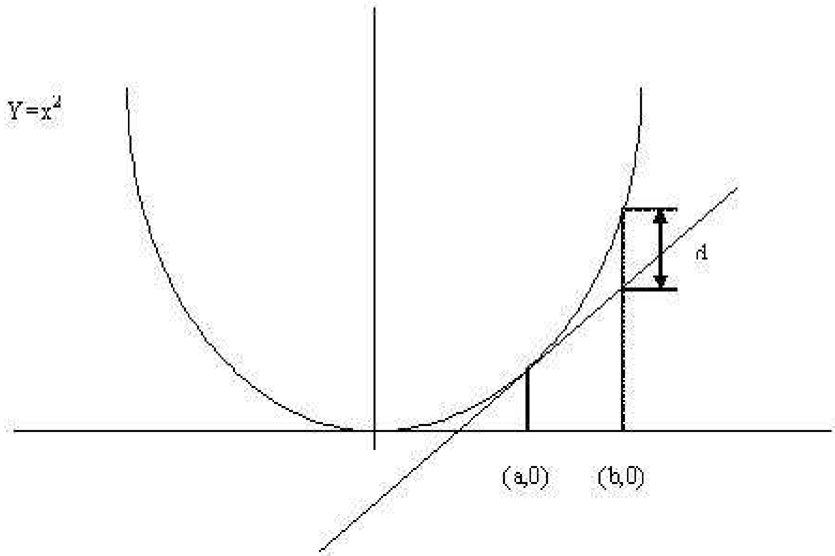
This method is used to create the voronoi diagram in d-dimension by computing the intersection of halfplanes in (d+1) dimension. Each point in the input set in d-dimension is mapped to a hafplane in (d+1) dimension. IN 2D, the point (u,v) is mapped to the tangent to the parabola z=x^2+y^2 at (u,v,u^2+v^2). Similarly, in 1D, the point a is mapped to the tangent y=2ax-a^2 to the parabola y=x^2 in 2D. Let b be any other point, and d defines the vertical distance of the tangent at (a,a^2) from the parabola y=x^2 at the point (b,b^2) as shown in the figure 1.

d = a^2-(2ab-b^2)=$(a-b)^2$ = $dist^2$(a,b) in 1D

Hence, we can see that d can be used as a measure to compare the distance between two points in 1D. Similar is true for higher dimensions and can be proved easily.

By using the above described method to construct the dual, the distinct input points can be mapped to distinct hyperplanes. These arrangement of hyperplanes can now be used to construct the voronoi diagram of the input points. As shown in the figure 2, if we consider the hyperplanes corressponding to points a and b and another point c in 1D, we find that c is closer to point a than b because the hyperplane defined by point a is closer to the parabola than the hyperplane defined by point b. Hence, in case of n hyperplanes, we can say a

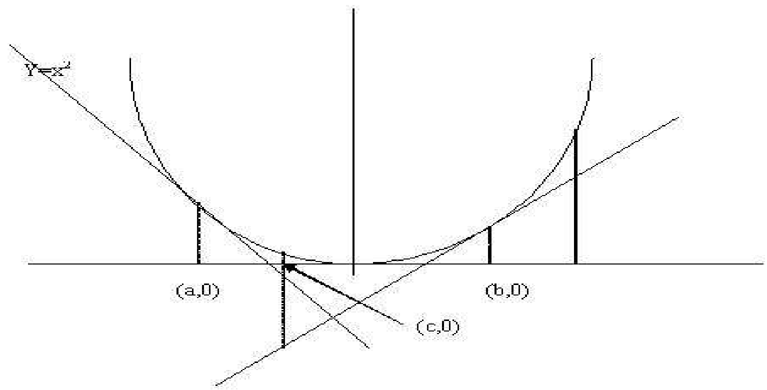
Figure 1: The point and its dual line



given point r is closest to the point which defines the hyperplane closest to the parabola at point (r,r^2). So, if we map the intersection of all the hyperplanes (with the direction of each hyperplane pointing upwards) on the d dimensional horizontal plane, than we get the required voronoi diagram of those n points. The above process can also be used to construct the higher order voronoi diagrams as well. Let us first define a k-order voronoi diagram. A region defined by a point p in k-order voronoi diagram is defined as the region in which every point has p as the k-th closest neighbour in the set of n points. It is quite obvious from the definition that the regions may not be connected in nature and we can have a set of disjoint regions associated with the point p. The voronoi diagrams, we have studied till now, belong to the first order voronoi diagrams as we take the closest neighbour as the criteria to define the regions. The (n-1)th voronoi diagram is also called as farthest -point voronoi diagram.

The second order voronoi diagram can be constructed by first removing the partial hyperplanes defining the first order voronoi diagram and then taking the intersection again as we did to find the first order voronoi diagram. Similarly, the k-th order voronoi diagram can be constructed by removing the partial hyperplanes defining 1,2,...(k-1)th voronoi diagrams. The planes corressponding to voronoi diagrams for the points in 1D are shown in the figure 3.

Figure 2: Distance of c from a is less than that from b



4 Dual Transform to construct the convex hull

This method is used to create the convex hull in d dimension by computing the intersection of half planes in d dimension. Each point in the input set in d-dimension is mapped to a hafplane. Let y=$x^2/2$ be a parabola as shown in the figure 4. Let the point P under consideration to construct the dual be located at (a,b). (a,a^2) is the point C on the parabola. Let d be the vertical distance between the point P and point C i.e. the distance CP. Then, the dual of the point P is the line which is parallel to the tangent to the parabola at point C and has the vertical distance d from the point C in the direction opposite to that of CP. The slope of the tangent is a. A point on the dual line is O with the coordinates as (a,a^2-b). Hence, the dual line which has slope a and passes through point O is y=ax-b. Hence, the point (a,b) is mapped to the line y=ax-b i.e. D((a,b)):= y=ax-b and D(y=ax-b):= (a,b). It is easy to verify that D(D(p))=p and D((l))=l.

This dual transform induces the following useful properties which are easy to prove.

- **Property 1:** Dual transform preserves incidence. This property states that if a point p lies on a line l, then D(l) lies on the line D(p).
- **Property 2:** Dual transform reverses the polarity. This property states that if a point p lies above line l(half space containing (0,inf), then D(l) lies above D(p).

To construct the convex hull, we take the dual of all the input points and find the intersection of the half planes pointing upwards. The corner points of the half spaces are taken

Figure 3: Ist, IInd and IIIrd order Voronoi diagrams shown in red, blue and green respectively.

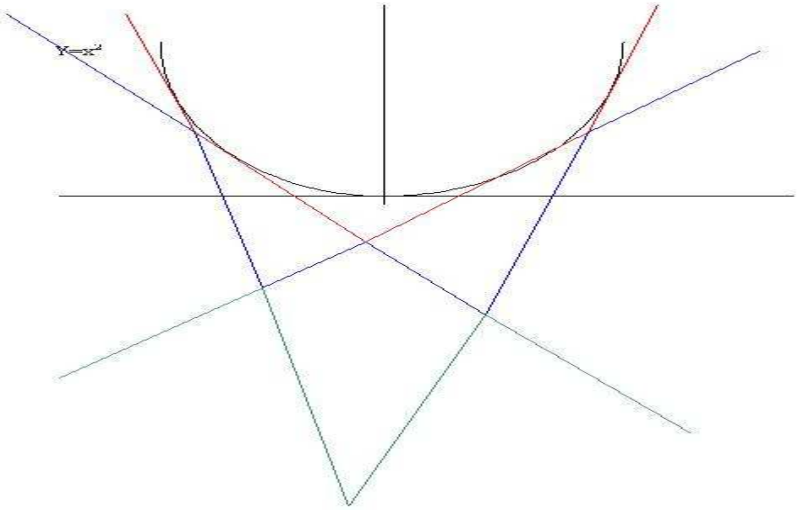
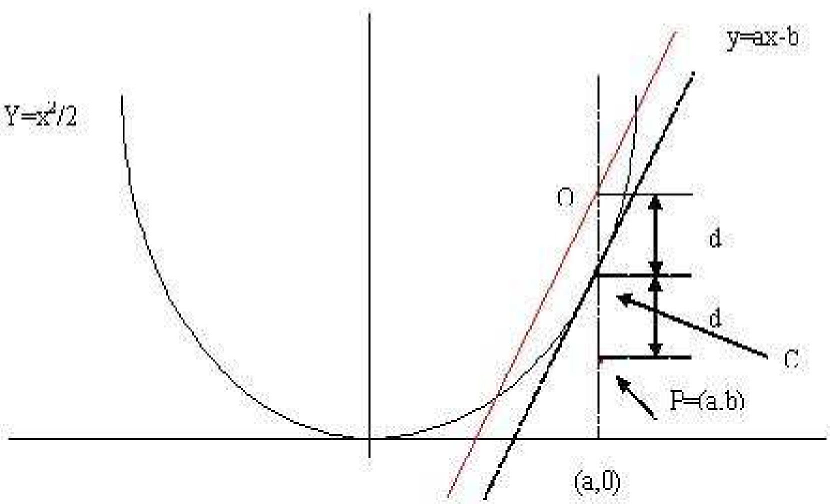


Figure 4: The point (a,b) and its dual line.



and their dual is found. Using the above properties, it can be shown that the figure thus obtained corressponds to the upper convex hull. Similarly, when we take the intersection of half planes pointing in the downward direction, we get the lower convex hull. Joining these two parts, we get the required convex hull.