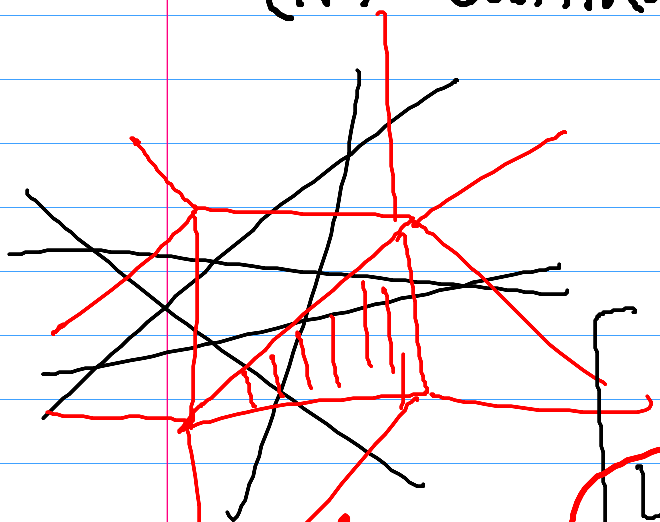
Lect. #32: Range Searching

Halfplane (space) range counting

(1/r) - cuttings



$L: \{l_1, \ldots, l_n\}$: n lines in $\mathbb{R}^2$

$\Xi = \{\Delta_1, \ldots, \Delta_s\}$:

A triangulation of $\mathbb{R}^2$ is called a (1/r)-cutting of L if every triangle intersects $\leq n/r$ lines of L

The wt of lines intersecting any Δ is $\leq W(L)/r$

Theorem: Given L and r, a (1/r)-cutting of L of size ~~$O(r^2)$~~ $O(r^d)$ can be computed in ~~$O(nr)$~~ $O(nr^{d-1})$ time.

Cutting Tree: Choose a constant r_0

H: current set of line

Δ: triangles

If $|H| \leq r_0$ stop

$|H| \geq r_0$

$\{\tau_1, \ldots \tau_s\} = \Xi_\Delta \left(\frac{1}{r_0}\right)$-cutting of H

H_i : set of lines intersecting τ_i

Make each τ_i a child of Δ

CuttingTree (H_i, τ_i) $\forall\, 1 \leq i \leq s$

n_{τ_i}:

for each τ_i store the # lines of H lying above τ_i.

Query procedure

Query(Δ, p)

If Δ is a leaf, return the # lines of H_Δ lying above p.

else

τ : triangle of Ξ_Δ that contains p

return n_τ + Query(τ, p)

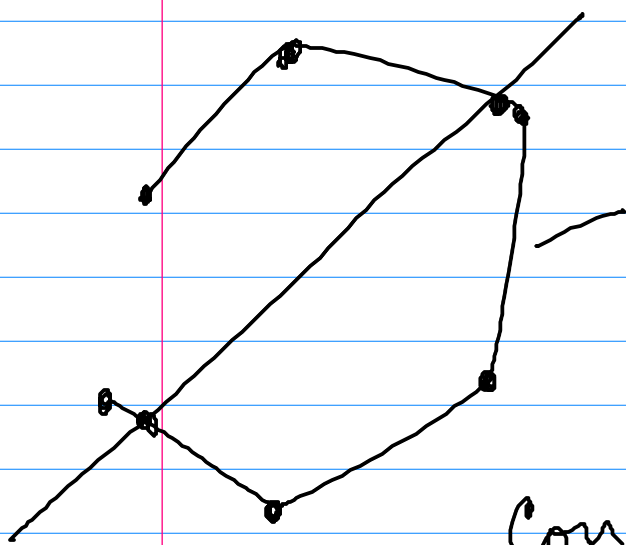
Time: $O(r_0^2 \log n)$

$S(n)$: max # leaves in a cutting tree built on n lines.

$$S(n) \le \begin{cases} 1 & n \le r_0 \\ c \cdot r_0^2 \cdot S(n/r_0) & n > r_0 \end{cases}$$

$$\begin{aligned} S(n) &\le c r_0^2 S(n/r_0) \\ &\le (c r_0^2)^i S(n/r_0^i) \\ &\le (c r_0^2)^{\log_{r_0} n} \\ &= c^{\log_{r_0} n} \cdot \underbrace{(r_0^2)^{\log_{r_0} n}}_{n^2} \\ & \underline{\underline{n^{\log_{r_0} c}}} \cdot \underline{n^2} \\ &\le n^{2+\varepsilon} \end{aligned}$$

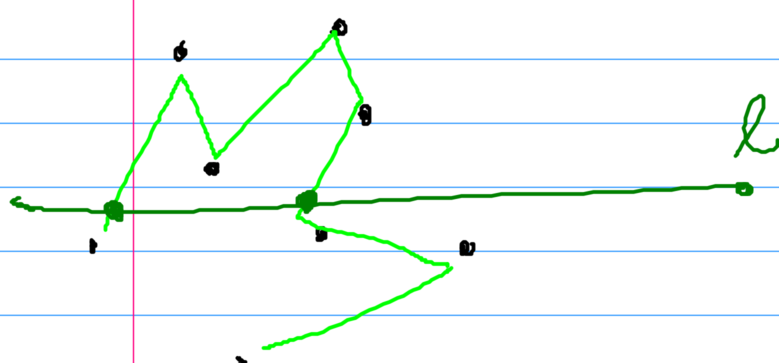
Linear size data structure



Spanning path

S: n points in $\mathbb{R}^2$

Construct a spanning path Π of S s.t. every line intersects as few edges of Π as possible.



Spanning path with small crossing number.

$\chi(\Pi, \ell)$: # crossing points

Crossing number of Π

$$\chi(\Pi) = \max_{\ell} \chi(\Pi, \ell)$$

$$\chi(S) = \min_{\Pi} \chi(\Pi)$$

What is the value of $\chi(S)$?

$$\chi(n) = \max_{\substack{|S|=n \\ S\subseteq \mathbb{R}^2}} \chi(S)$$

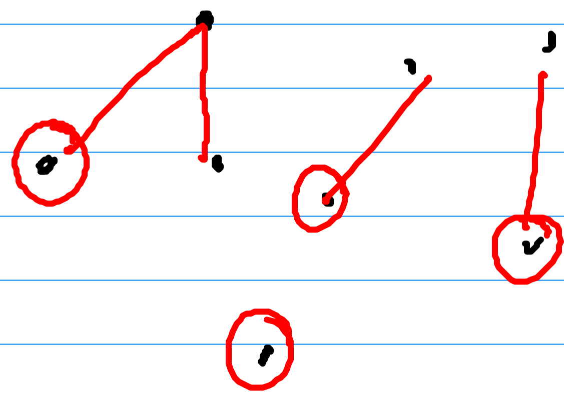
$$\Omega(\sqrt{n}) = \chi(n) = O(\sqrt{n})$$

S: set of n points

L: set of m lines

$w: L \longrightarrow \mathbb{N}$

$w(\ell) = 1 \;\; \forall \ell \in L$



i^{th} step:

S_i: one endpoint from each connected component

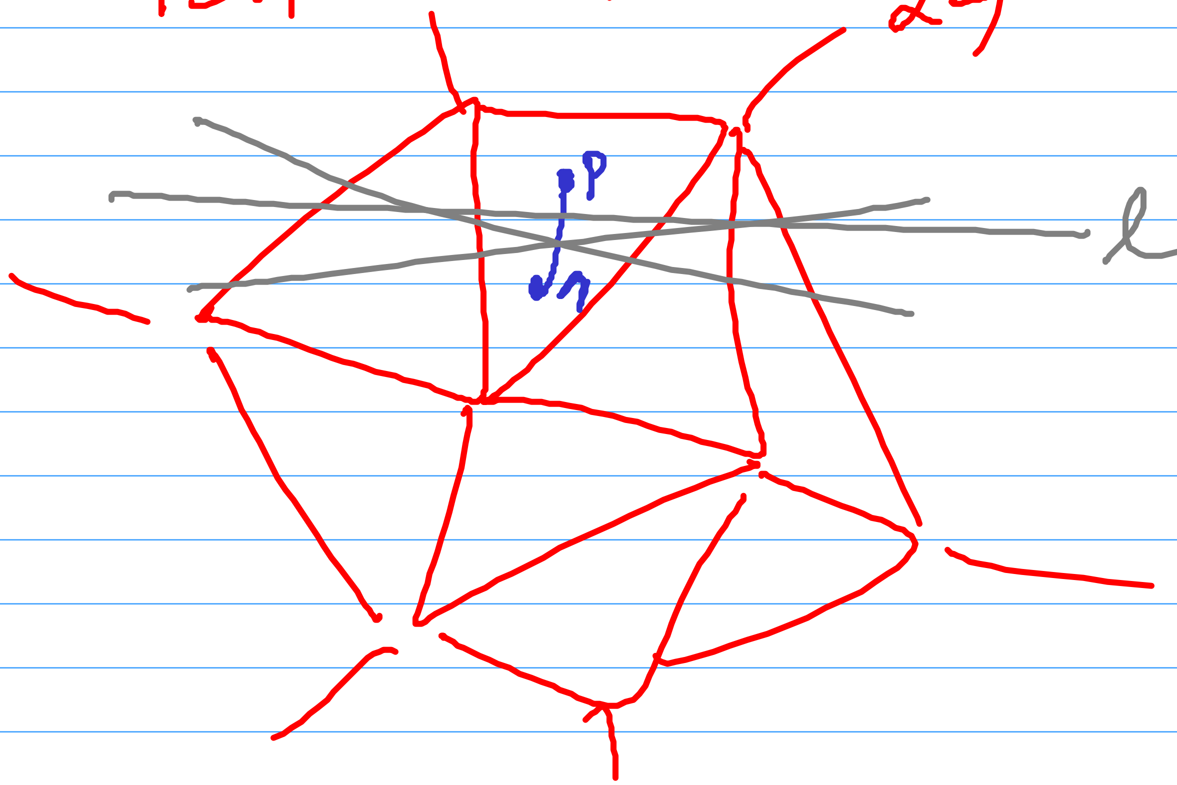
$n_i = |S_i| = n-i+1$

Choose a constant α

Compute a $\underbrace{(\alpha/\sqrt{n_i})}_{(1/r_i)}$-cutting Ξ_i of L

$r_i = \sqrt{n_i}/\alpha$

$|\Xi_i| = O(r_i^2) = O\left(\frac{n_i}{\alpha^2}\right) \le n_i - 1$



$\exists$ a $\Delta \in \Xi_i$ that contains 2 points $p, q \in S_i$

Add (p,q) to Π.

For each line ℓ (h) in L that intersects (p,q)

set $w(\ell) = 2 \cdot w(\ell)$

W_i : $w(L)$ after i^{th} iteration

$$W_0 = m$$

$$W_{i+1} \leq W_i + \frac{W_i}{r_i}$$

$$= W_i \left(1 + \frac{\alpha}{\sqrt{n-i+1}}\right)$$

$$W_{n-1} \leq W_0 \cdot \prod_{i=0}^{n-1} \left(1 + \frac{\alpha}{\sqrt{n-i+1}}\right)$$

$$= W_0 \prod_{j=1}^{n} \underbrace{\left(1 + \frac{\alpha}{\sqrt{j}}\right)}$$

$$\leq W_0 \prod_{j=1}^{n} e^{\frac{\alpha}{\sqrt{j}}}$$

$$\boxed{W_{n-1} \leq m \cdot \exp(\alpha \cdot \sqrt{n})}$$

$$\vee\!\!|$$

$$2^k$$

$$2^k \leq m \cdot \exp(\alpha \sqrt{n})$$

$$k = O(\sqrt{n})$$