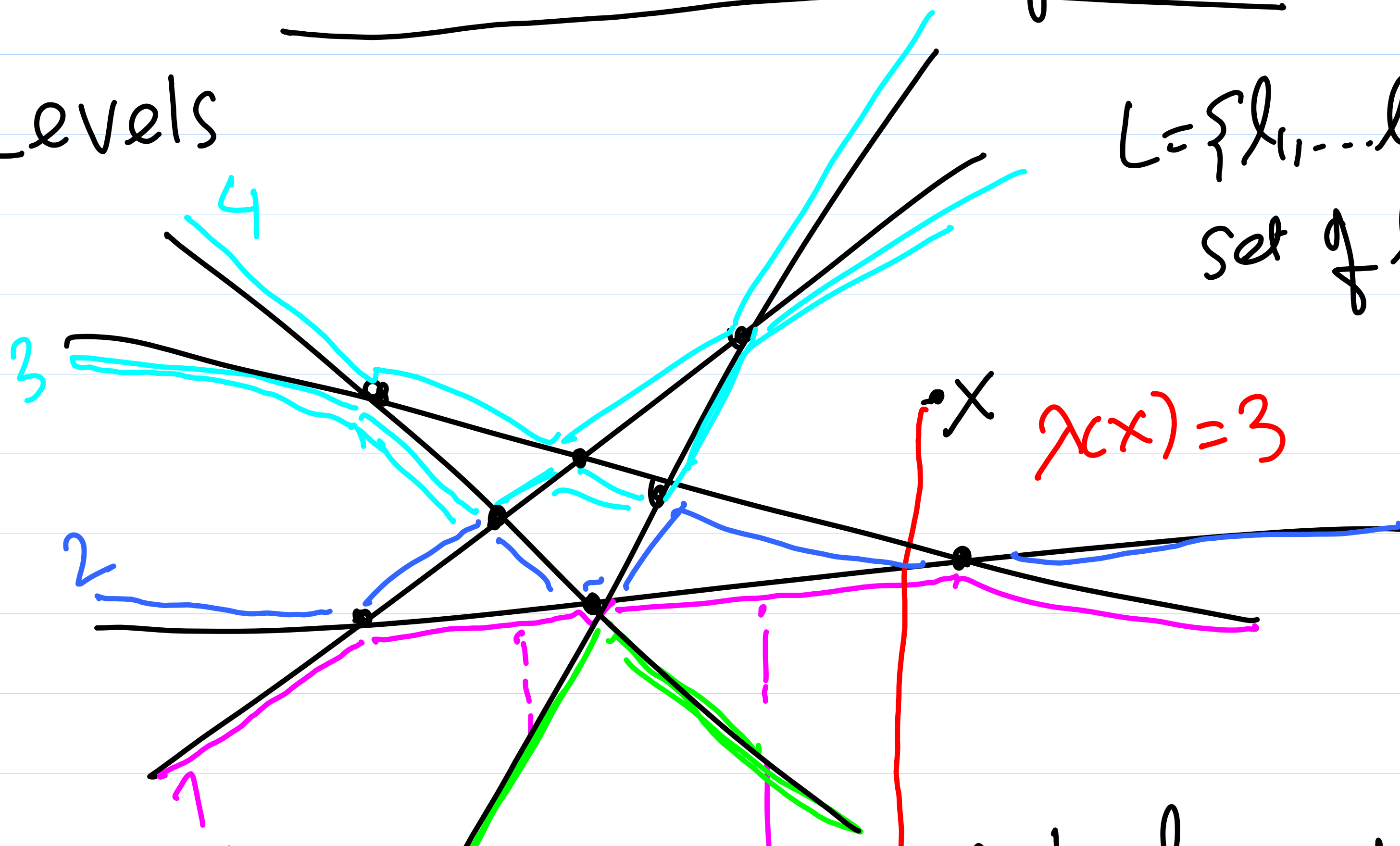
Lecture 25: Arrangements

Levels

$L = \{l_1, \dots, l_n\}$: set of lines



$\lambda(x)$: Level of x : # lines of L lying below x

k-level $A_k(L)$: set of edges of $A(L)$ whose level is k.

$Q_k(L)$: # vertices on k-level ($A_k(L)$)

$$Q_k(n) = \max_{|L|=n} Q_k(L)$$

average size of $A_k(L) = O(n)$

What's the worst case complexity of $A_k(L)$?

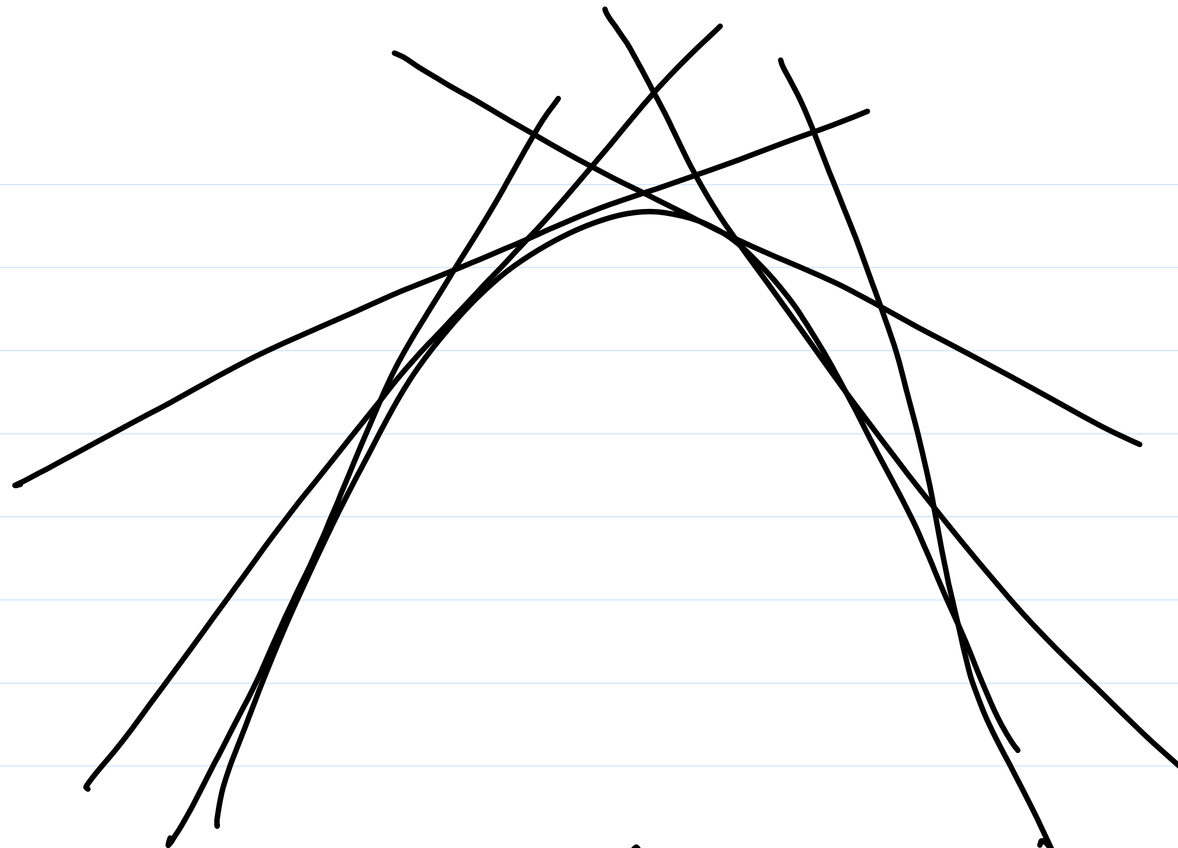
$Q_k(n) = O(nk^{1/3})$

$\Omega(n \log k)$

$A_0(L)$ (level 0) : lower envelope

$A_{n-1}(L)$ (level $n-1$) : upper envelope

H : set of Voronoi planes in $\mathbb{R}^3$



What does level k $A_k(H)$ correspond to?

($\leq k$-level) $A_{\leq k}(L)$: set of edges of $A(L)$ whose level is $\leq k$.

$Q_{\leq k}(L)$: # vertices of $A(L)$ whose level is $\leq k$

$\checkmark$ $Q_{\leq k}(L) = \sum_{j=0}^{k} Q_j(L)$

Theorem: $Q_{\leq k}(L) = \Theta(nk)$

Proof: fix a parameter $0 \leq p \leq 1$

Choose each line of L with prob. p.

R: set of chosen lines.

$E[|R|] = pn$

$A_0(R)$: level 0 of R.

$V_k(L)$: set of vertices of $A(L)$ whose level is k.

$Q_k(L) = |V_k(L)|$

$$E[Q_0(R)] = \sum_{v \in A(L)} \Pr[v \in V_0(R)]$$

$$= \sum_{j=0}^{n-1} \sum_{v \in V_j(L)} \Pr[v \in V_0(R)]$$

$$= \sum_{j=0}^{n-1} \sum_{v \in V_j(L)} p^2 (1-p)^j$$

$$\geq \sum_{j=0}^{k} p^2 (1-p)^j Q_j(L)$$

$$\geq \sum_{j=0}^{k} p^2 (1-p)^k Q_j(L)$$

$$\geq p^2 (1-p)^k Q_{\leq k}(L)$$

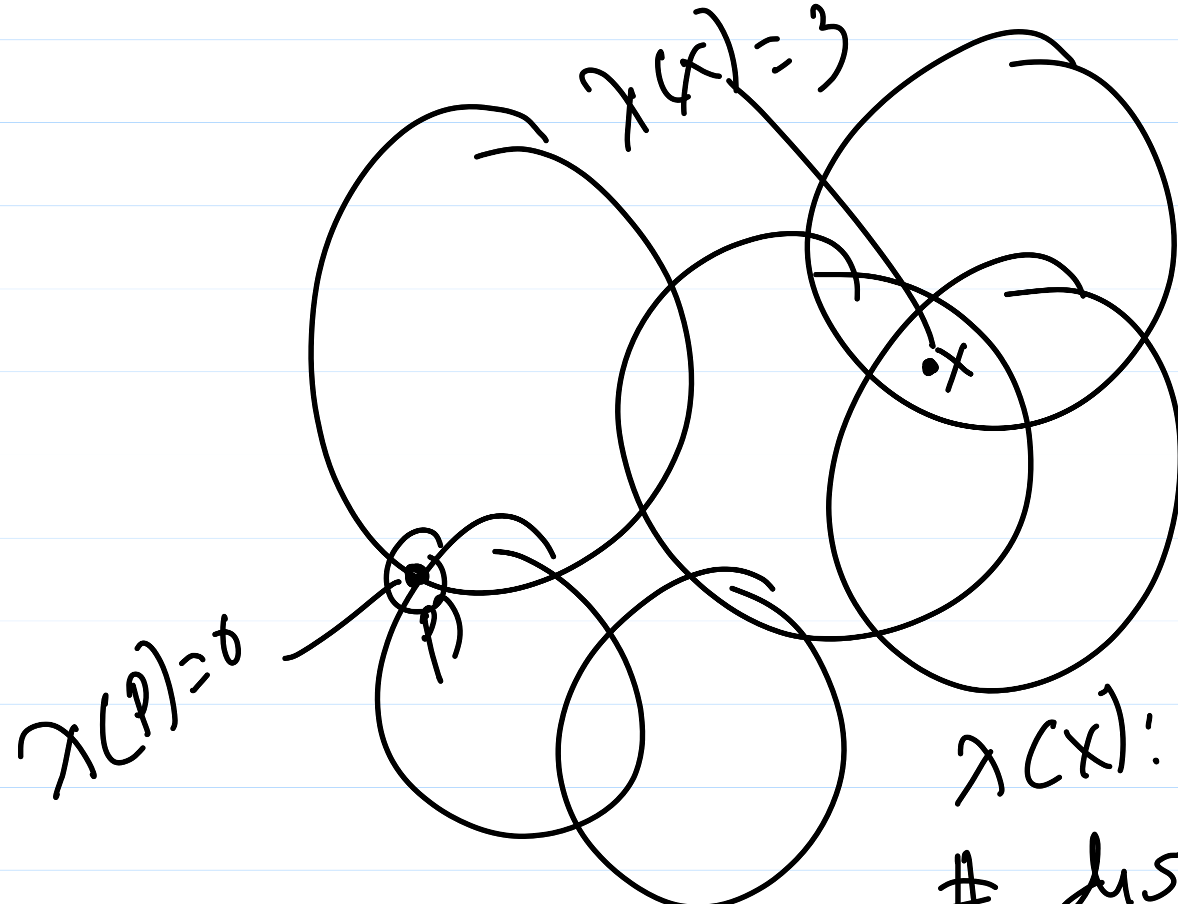
$$\boxed{Q_{\leq k}(L) \leq E[Q_0(R)] / p^2 (1-p)^k}$$

Set $p = 1/k$

$$\underline{Q_{\leq k}(L)} \leq \frac{k^2}{(1-\frac{1}{k})^k} \cdot \frac{n}{k}$$

$$\leq e \cdot nk = O(nk)$$

$$\boxed{Q_{\le k}(n) \le k^2 \cdot e\, Q_0(n/k)}$$



$D = \{D_1, \ldots, D_n\}$: n disks

$x \in \mathbb{R}^2$

$\lambda(x)$: level of x

\# disks that contain x in their interior.

$A_0(D)$: Boundary of $\bigcup_{i=1}^{n} D_i$

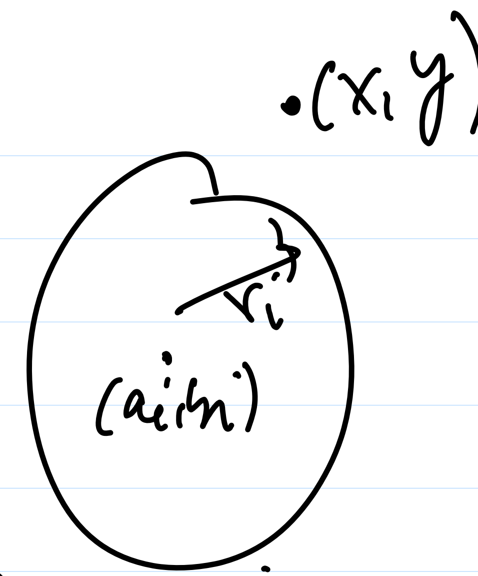
$A_k(D)$: k-level

$Q_k(D)$, $Q_{\le k}(D)$

$\boxed{\textbf{Lemma 1:} \quad Q_0(D) = O(n)}$

Lemma 2: $Q_{\le k}(D) = O(nk)$

$$D = \{D_1, \ldots, D_n\}$$



$$D_i = (a_i, b_i), r_i$$

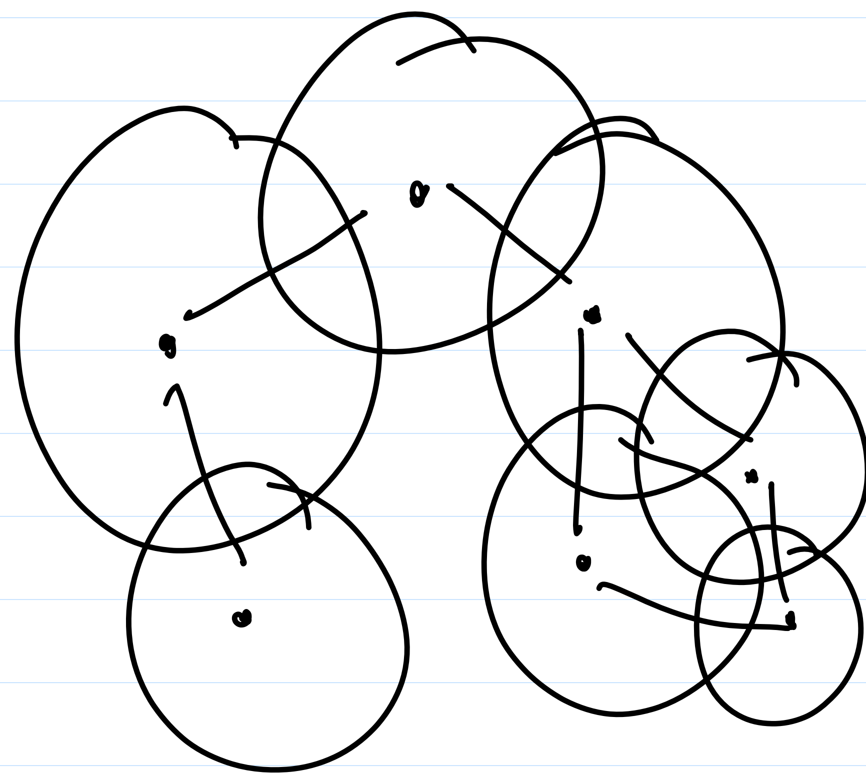
$$(x - a_i)^2 + (y - b_i)^2 \geq r_i^2 \quad \forall i$$

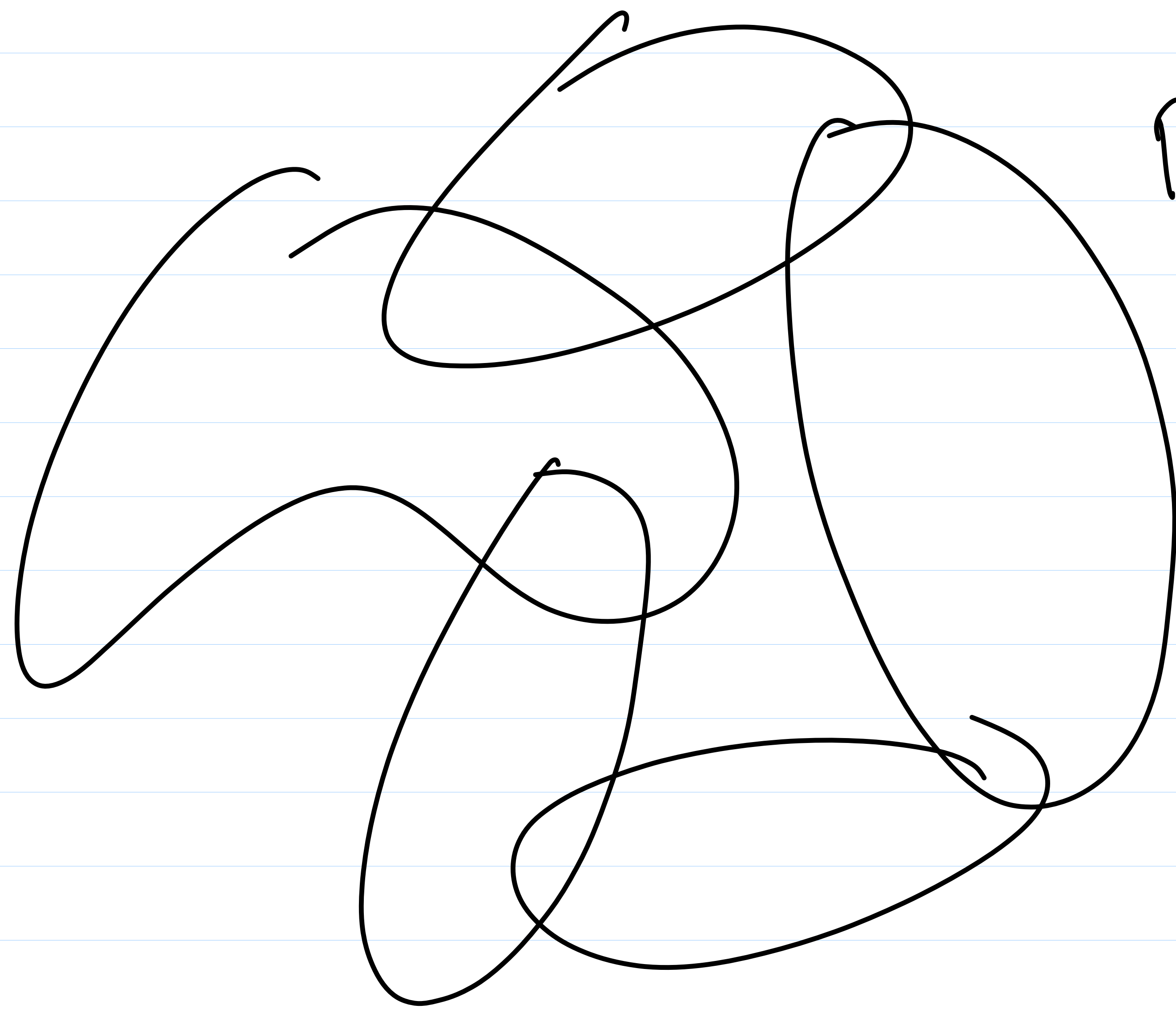
$$\forall (x, y) \notin \bigcup_{i=1}^{n} D_i$$

$$\underbrace{x^2 + y^2}_{=z} - 2a_i x - 2b_i y + a_i^2 + b_i^2 \geq r_i^2$$

$$h_1 : z \geq 2a_i x + 2b_i y + r_i^2 - a_i^2 - b_i^2 \quad \forall i$$

$\bigcap_{i=1}^{n} h_i$ has $O(n)$ vertices





Pseudo disks