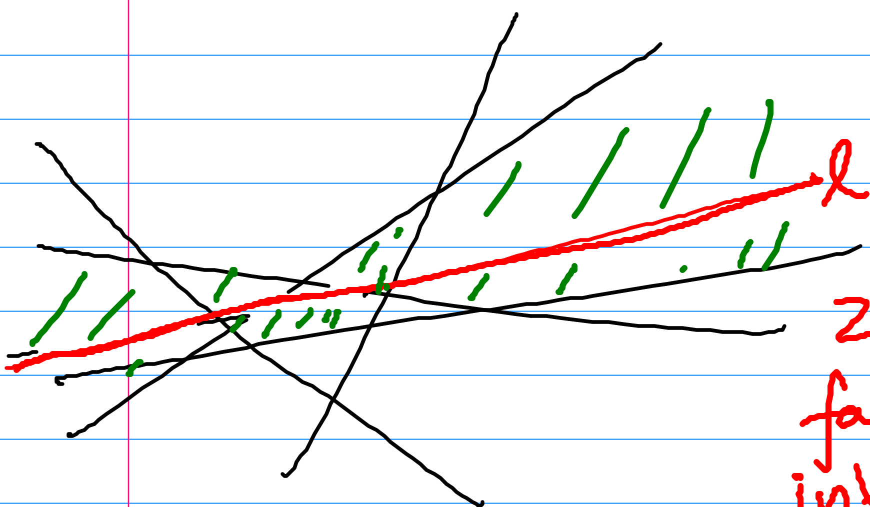
Lecture 24: Arrangements

Zone Theorem:

$L = \{\ell_1, \dots, \ell_n\}$: n lines in $\mathbb{R}^2$



f: face of $A(L)$

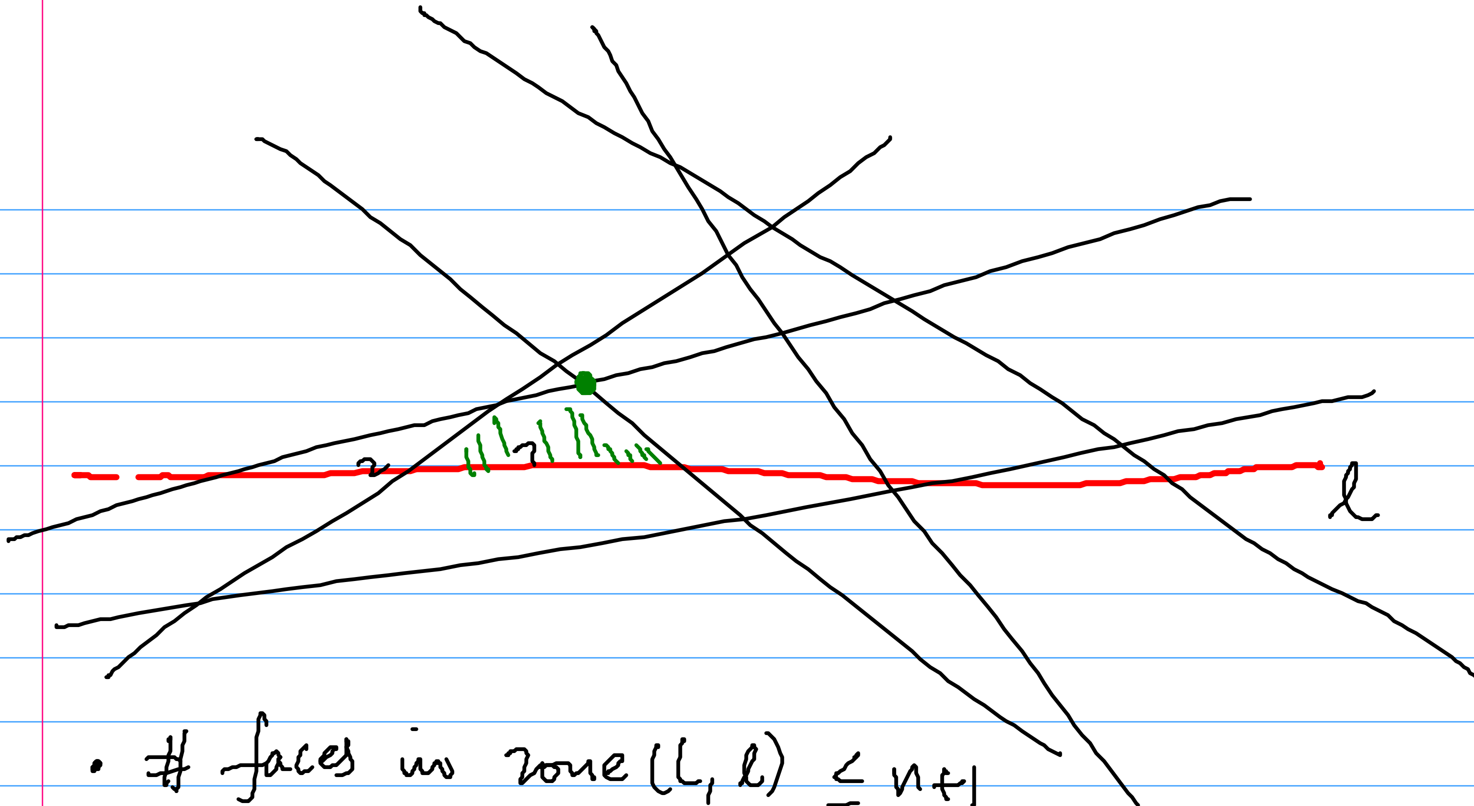
n_f: # edges in f

Zone(L, ℓ): set of faces of $A(L)$ that intersect ℓ.

$$\mu(L, \ell) = \sum_{f \in Zone(L,\ell)} n_f$$

$$\mu(n) = \max_{|L|=n} \mu(L, \ell)$$

Theorem: $\mu(n) = O(n)$



- # faces in zone(L, ℓ) $\leq n+1$
- Boundary of each face divided into left & right chains by its topmost vertex.
- Charge the top edges of left & right chains to that face. $\sim 2n$ edges

- Each unmarked edge on the left chain is charged to the line of L on which it lies
- Do the same for the unmarked edges on the right chains

Lemma: Each line of L is charged at most once by the unmarked edges on the left chains. The same is true for right chains.

$$\Downarrow$$

$$\mu(L, l) = O(n) \qquad \square$$

H: $\{h_1, \ldots, h_n\}$: set of hyperplanes in $\mathbb{R}^d$

$A(H)$: arrangement of H. $O(n^d)$

Zone(H, h): Set of faces of $A(H)$ that intersect h.

$\mu(H, h)$: Complexity of zone(H, h)
vertices

$$\mu(n, d) = \max_{|H|=n} \mu(H, h)$$

Theorem: $\mu(n, d) = O(n^{d-1})$

Lemma: $\displaystyle\sum_{f\in A(L)} n_f^2 = O(n^2)$

Pf:

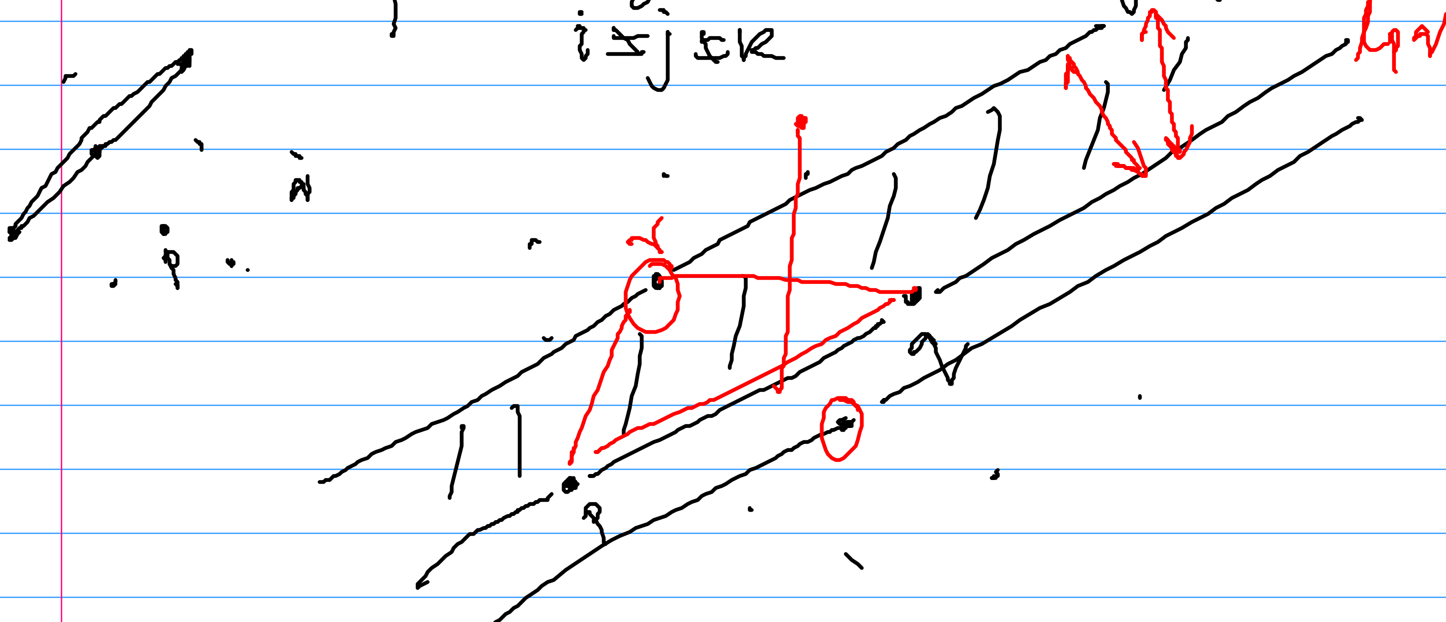
$$\sum_{f\in A(L)} 2\binom{n_f}{2} + n_f$$

$$= \sum_{f\in A(L)} 2\binom{n_f}{2} + O(n^2)$$

Claim: $\displaystyle\sum_{f\in A(L)} 2\binom{n_f}{2} = \sum_{\ell\in L} \mu(L\setminus\{\ell\}, \ell)$

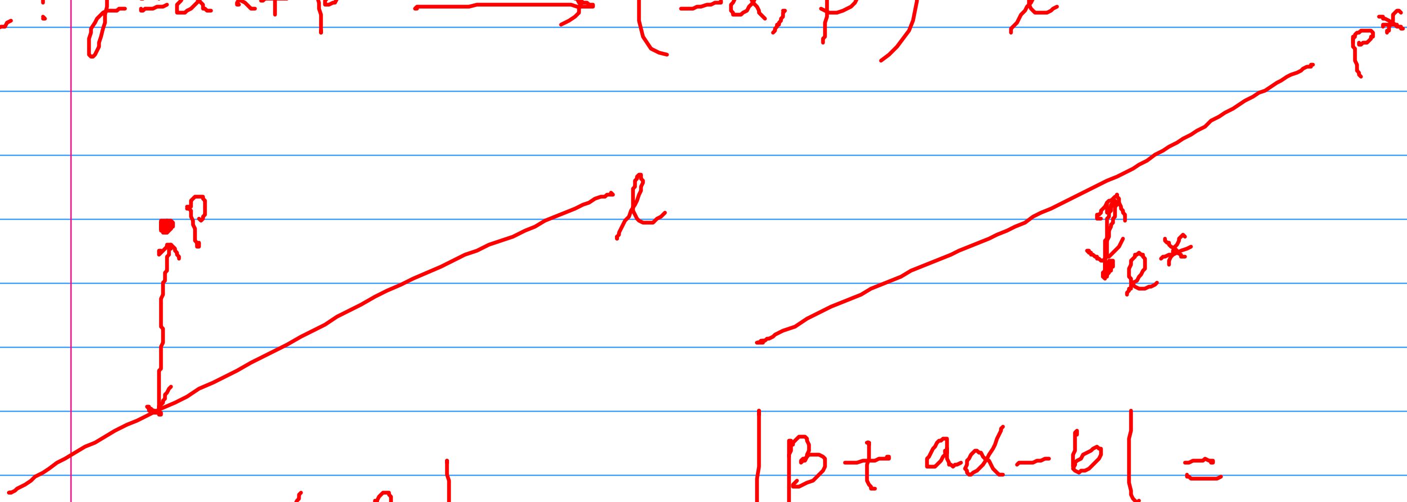
Prob: $S: \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$

compute $\displaystyle\operatorname*{argmin}_{i\neq j\neq k} \text{Area}\,\triangle p_i p_j p_k$



$p : (a, b) \longrightarrow y = ax + b \quad p^*$

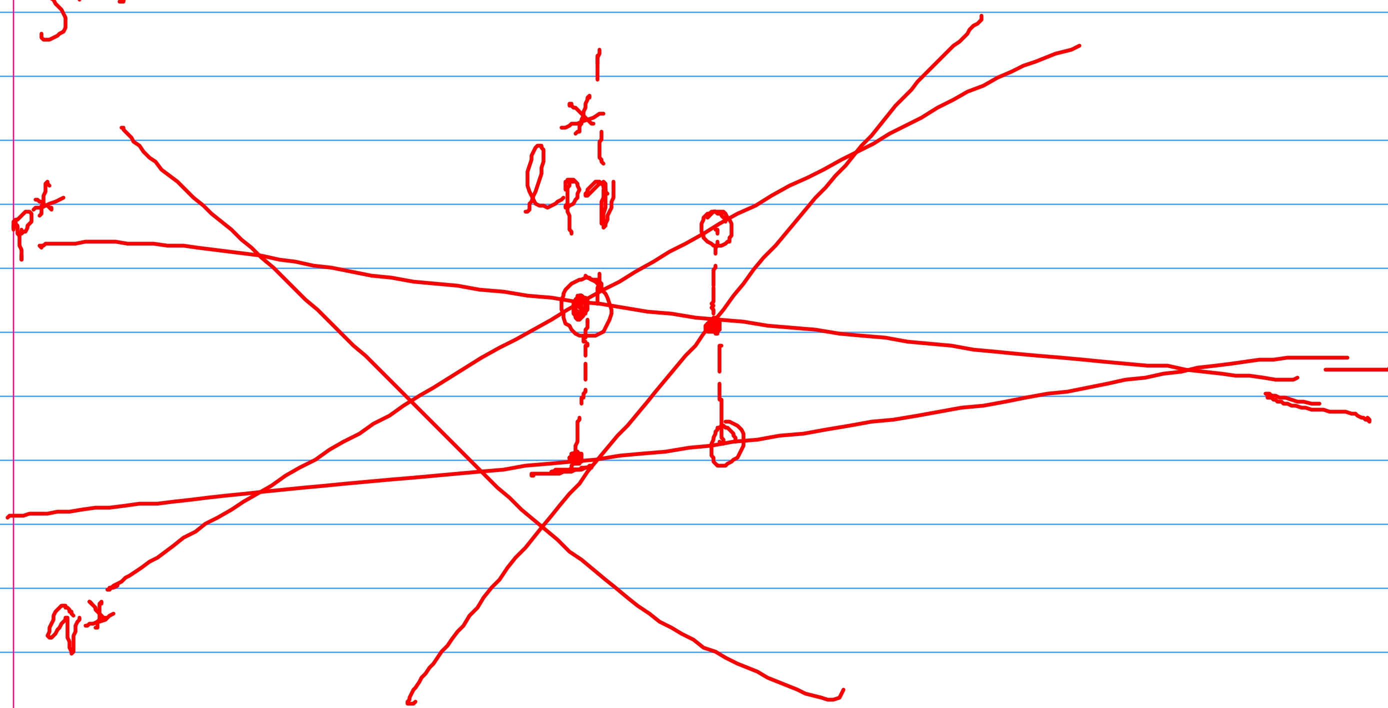
$\ell : y = \alpha x + \beta \longrightarrow (-\alpha, \beta) \quad \ell^*$



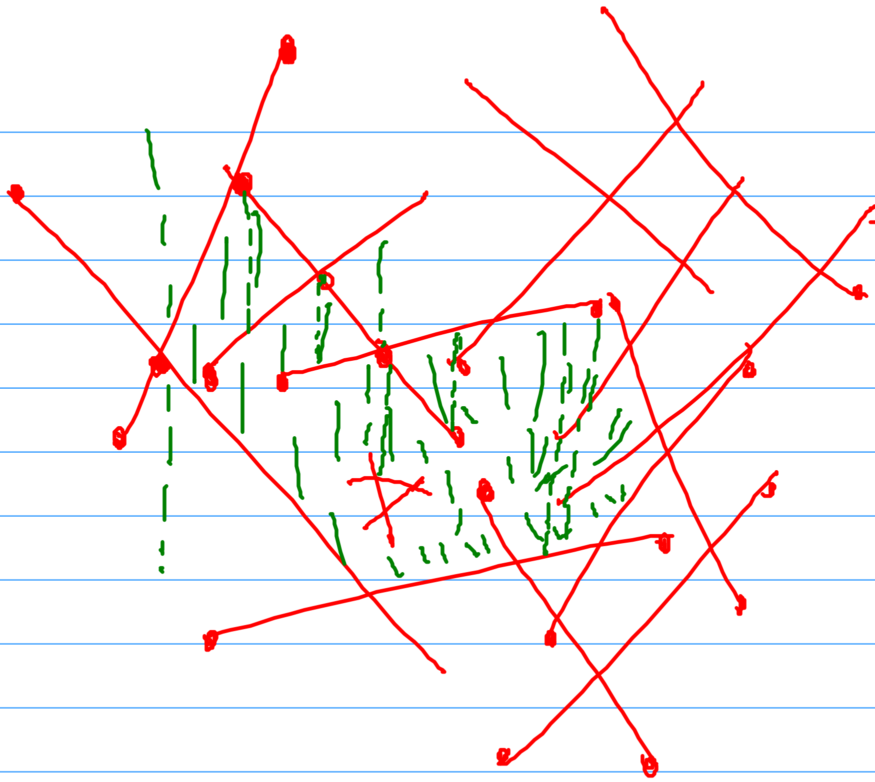
$|b - a\alpha - \beta|$

$|\beta + a\alpha - b| = |b - a\alpha - \beta|$

S^*



Compute $A(S^*)$



S: set of n segments

Complexity of a face in A(S) is

$\Theta(n\alpha(n))$ $\alpha(n)$: inverse Ackermann function.

Vertical decomposition of the arrangement.

Randomized algorithm

$O(n\log n + k)$ expected time

k: # vertices.