CSL 852, Computational Geometry: Practice Problems

1. **Dynamic Convex hull** In many applications, we are required to maintain the convex hull of a set of points that could be changing over time, i.e., points can be inserted or deleted. Try to construct cases where a single insertion/deletion can lead to large changes in the size of the hull. Therefore, we can try to maintain an *implicit* representation of the convex hull so that we can report it in time $O(h)$ where h is the size of the present hull.

   Describe all the details of maintaining a convex hull under a sequence of insertions and deletions such that the cost of each insert/delete operation is bounded by $O(\log^2 n)$ and the space of the data structure is $O(n)$. Moreover the data structure should also support query of the type - "Is a given point inside the convex hull?" in $O(\log n)$ time.

   Hint: Devise a scheme based on a primary tree structure whose leaves correspond to the existing points and each internal node stores a data structure representing the upper/lower hull of all the points in the subtree rooted at that point. This data structure must support fast *join* and *split* operation also known as a concatenable queue. Each internal node contains a bridge (tangent) of the two linearly separable convex hulls of its two subtrees - it does not contain an explicit description of the convex hull to save space. Recall that such a bridge can be found in $O(\log n)$ steps. Argue that any insertion or deletion of a point can affect bridges along a single root-leaf path of this tree and the corresponding $O(\log n)$ bridges can be recomputed on the fly.

2. *Convex layer* The i-th convex layer if a given set S of points is the convex hull of the (remaining) set of points after deleting the points in layers 1 to $i-1$. The *first* layer is the convex hull itself. Describe an $O(n polylog(n))$ algorithm to determine all the layers.
   Hint: Can you apply the result of the previous problem ?

3. Given two sets A and B such that $|A|+|B| = n$, prove an $\Omega(n \log n)$ bound to determine if $A \cap B = \Phi$.
   Hint: Consider an alternating sequence of points and argue that each order type corresponds to a component in the solution space.

4. Describe all the details of the linear time algorithm (especially the correctness) for triangulating a 1-sided monotone polygon.

5. Describe all the details for subdividing a simple polygon into 1-sided monotone polygon in $O(n \log n)$ steps.

6. In Kirkpatrick's decomposition, the depth of the data structure (the levelled Directed acyclic subgraph) depends of the fraction of constant degree vertices that we an eliminate in each phase - we showed that this fraction α is at least 1/25. Using more careful reasoning show that $\alpha \geq \frac{4}{70}$ by optimising the value of degree k to maximise the number of vertices eliminated.

7. Prove the following about a set of points (no three are collinear and no four points are cocircular).
   (i) The circumcircle of a delaunay triangle doesn't contain any other input point.
   (ii) A pair of points is an edge of Delaunay triangulation iff there is a circle passing through these points that is empty.

8. The *Relative Neighbourhood Graph* (RNG) of a set of points is one where there is an edge between points p_i and p_j iff the $lune(i,j)$ does not contain any other points inside it. $Lune(i,j)$ is a region that is common to the two circles with radius $d(i,j)$ and centers p_i and p_j. Show that for any given set of points S

$$\text{Euclidean MST (S)} \subseteq \text{RNG (S)} \subseteq \text{DT (S)}$$