COV 886, Problem Sheet 2

1. Given a range space $S = (X, R)$ with VC dimension $d$
   (i) What is the VC dimension of $(X, \bar{R})$ where $\bar{R} = \{r \mid X - r \in R\}$, i.e., the complement space.

2. Consider the range space $S = (X, \mathcal{R})$ where $X$ is the set of points in the Euclidean $d$-dimensional space $\mathbb{E}^d$ and $\mathcal{R}$ is the set of closed half-spaces in $\mathbb{E}^d$.
   (i) What is the VC dimension of $S$?
   Hint: You may want to make use of the following result called Radon’s theorem. For a given set of $d + 2$ points in $\mathbb{E}^d$, there exists a disjoint partition of these points, say $C, D$ such that $CH(C) \cap CH(D) \neq \emptyset$ where $CH(\cdot)$ denotes the convex hull of a set of points.
   (ii) Is it easier to bound the shattering dimension of $S$? What bound does it yield on the VC dimension?

3. If $R_1$ and $R_2$ are $\epsilon$-samples of $P_1$ and $P_2$ where $P_1$ and $P_2$ are disjoint, then $R_1 \cup R_2$ is an $\epsilon$ sample of $P_1 \cup P_2$.

4. For a range space with discrepancy bounded by $\log^c n$ (polylog) rederive the bound for $\epsilon$ sample.

5. Prove the following theorem using discrepancy. Let $(X, R)$ be a range space with shattering dimension $d$, where $|X| = n$, and let $0 < \varepsilon < 1$ and $0 < p < 1$ be given parameters. Then one can construct a set $N \subset X$ of size $O(\frac{d}{\varepsilon^2p} \log \frac{d}{\varepsilon p})$ such that, for each range $r \in R$ of at least $pn$ points, we have

$$|\frac{|r \cap N|}{|N|} - \frac{|r \cap X|}{|X|}| \leq \varepsilon \frac{|r \cap X|}{|X|}$$

Then $N$ is called a relative $(p, \varepsilon)$-sample for $(X, R)$. 