Computational Geometry Lecture 36

Topic : ε-nets & VC dimension contd.

<u>Shatter function</u> : For a range space $S = (X, R)$, $\pi_S(m)$ is a shatter function if $\pi_S(m) = \max_{\substack{B \subset X \\ |B| = m}} |R_{|B}|$ $= \{r \cap B \mid r \in R\}$

Shattering dimension of S is the smallest d, such that $\pi_S(m)$ is $O(m^d)$ for all m.

$= c \cdot m^d$

Sauer's Lemma: If $S=(X,R)$ has VC dimension δ, the $|R|$ is bounded by $\underline{\underline{n^\delta}}$ for $|X|=n$

More precisely, it is bounded by

$$G_\delta(n) = \sum_{i=0}^{\delta} \binom{n}{i}$$

Remark: $G_\delta(n) = G_{\delta-1}(n-1) + G_\delta(n-1)$

Proof by induction on δ, n

Suppose true for all values of n till $\delta-1$ and true for all values of δ till $n-1$.

$S = (X, R)$, for a fix $x \in X$, we define

$R_x : \{ r \in R \mid x \notin r \text{ and } r \cup \{x\} \in R \}$

$R - x : \{ r - x \mid r \in R \}$

$$|R| = \underbrace{|R_x|}_{G_{\delta-1}(n-1)} + \underbrace{|R-x|}_{\text{one element less}}$$

Observation:

V.C. dimension is $\delta-1$

From I.H. $G_\delta(n-1)$

There exists some set Z $|Z| = \delta-1$ and Z can be shattered, i.e. all subsets of Z belong to R_x

Then all subsets of $Z \cup \{x\}$ can be shattered in R

(Note $r' \in R_x \Rightarrow r' \in R$ and $r' \cup \{x\} \in R$) by defn of R_x

$$|R_x| \le G_{\delta-1}(n-1)$$

$$\text{So } |R| \le G_{\delta-1}(n-1) + G_\delta(n-1)$$

$$= G_\delta(n)$$

If $S = (X, R)$ has VC dimension δ, then shattering dimension $\leq \delta$

<u>Claim</u> : If shattering dimension of S is d, then the VC dimension of S is $O(d \log d)$

<u>proof</u> : If $N \subseteq X$ is the largest set shattered by S, then $|N| = \delta$ (VC)

$$2^{\delta} \leq c \cdot \delta^{d}$$

size of N

$$\Rightarrow \quad \delta \leq O(d \log d)$$

Dual Range Space

$S = (X, R)$ is (primal) range space

then $S^* = \{R, X_p \mid p \in X\}$ is a dual range space where

$$X_p = \{r \in R \mid p \in r\}$$

<u>Observation</u> : $(S^*)^* = S$

$\Rightarrow$ If S has V.C. dimension δ, then S^* has V.C. dimension $\leq 2^\delta$

$\Leftarrow$ If S has dual shattering dimension δ, then S^* has V.C. dimension $O(\delta \log \delta)$ and therefore $(S^*)^* = S$ has V.C. dimension $2^{\delta \log \delta}$