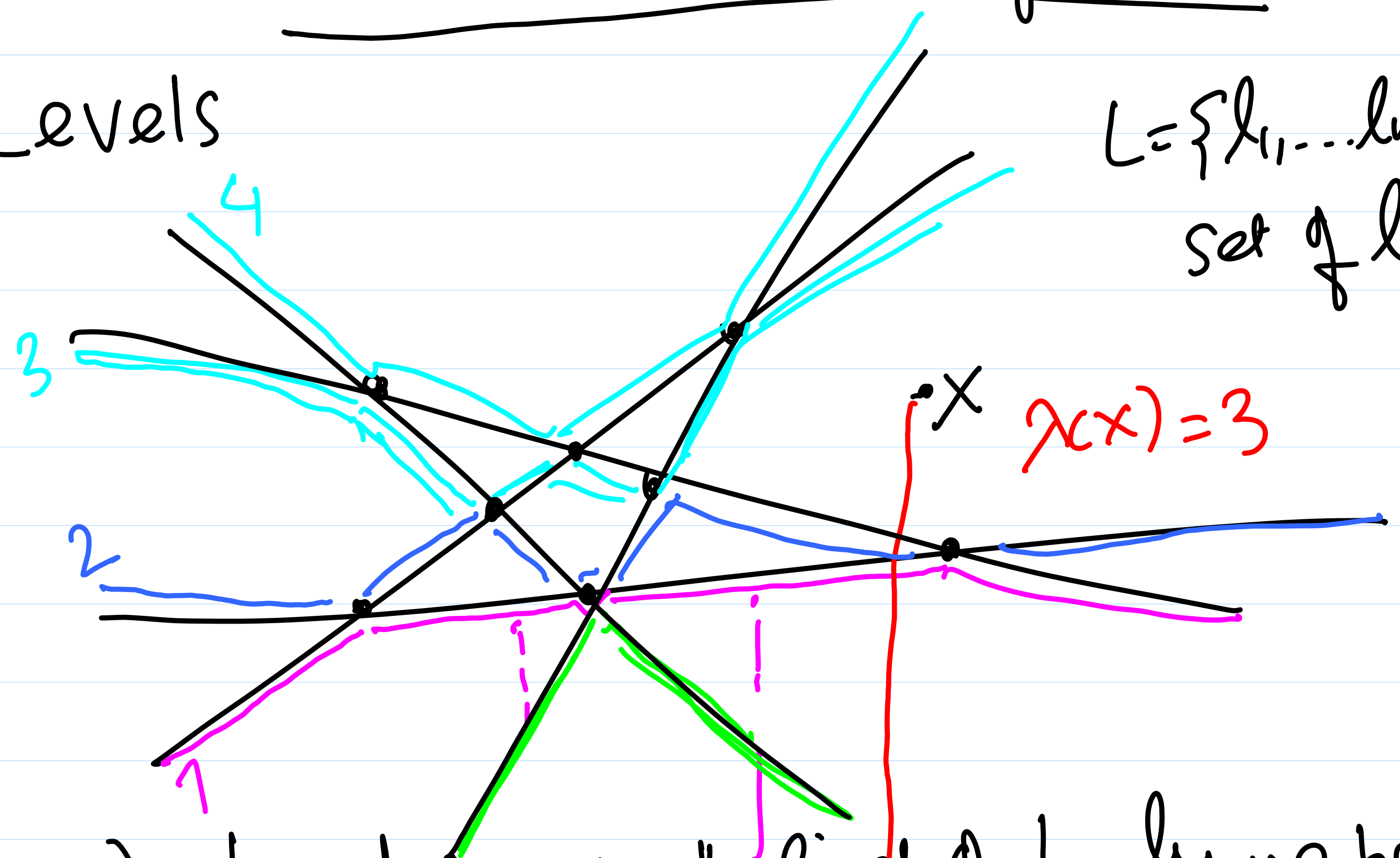
Lecture 25: Arrangements

Levels

$L = \{\ell_1, \dots, \ell_n\}$: set of lines



$\lambda(x)$: Level of x : # lines of L lying below x

k-level $A_k(L)$: set of edges of $A(L)$ whose level is k.

$Q_k(L)$: # vertices on k-level ($A_k(L)$)

$$Q_k(n) = \max_{|L|=n} Q_k(L)$$

average size of $A_k(L) = O(n)$

What's the worst case complexity of $A_k(L)$?

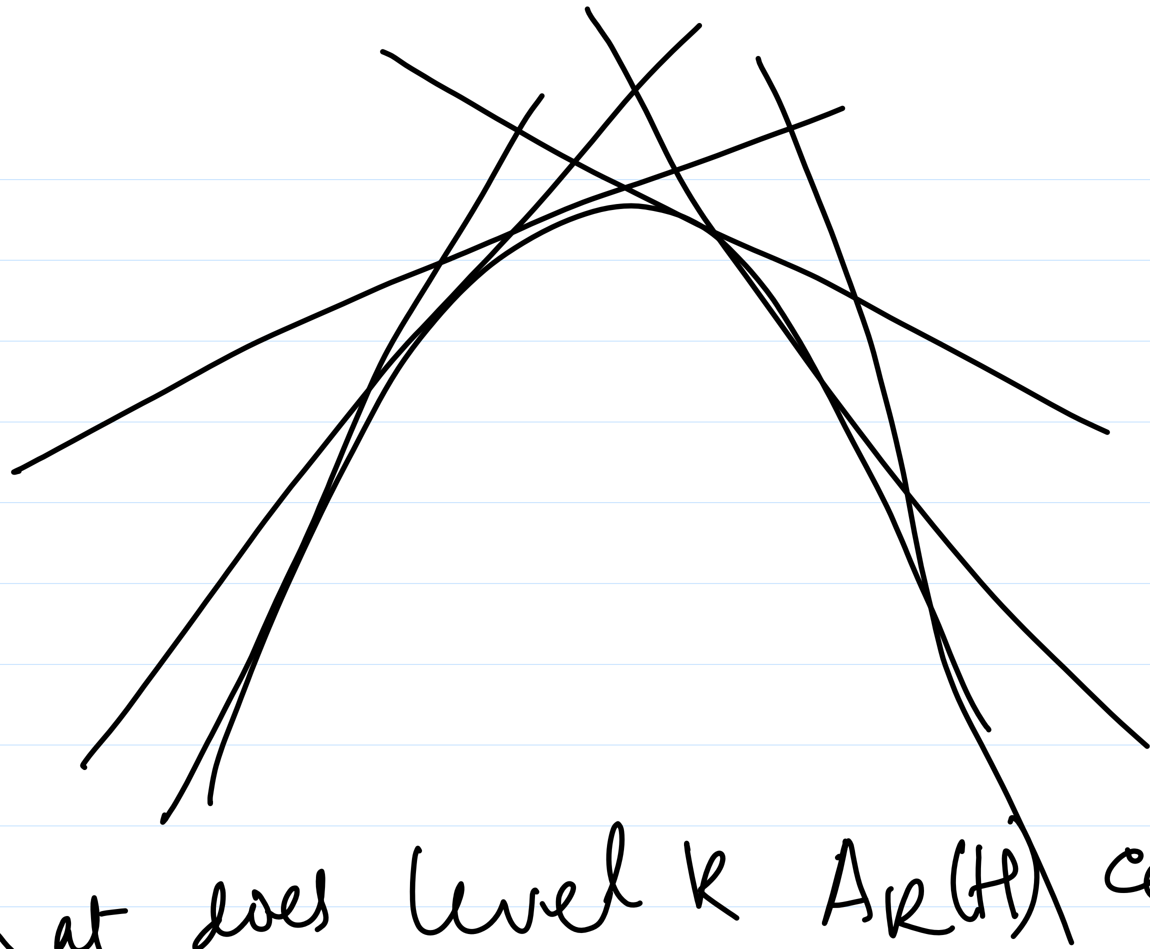
$$Q_k(n) = O(n k^{1/3})$$

$$\Omega(n \log k)$$

$A_0(L)$ (level 0) : lower envelope

$A_{n-1}(L)$ (level $n-1$) : upper envelope

H': set of Voronoi planes in $\mathbb{R}^3$



What does level k $A_k(H)$ correspond to?

($\leq k$-level) $A_{\leq k}(L)$: set of edges of $A(L)$ whose level is $\leq k$.

$Q_{\leq k}(L)$: # vertices of $A(L)$ whose level is $\leq k$

$$\checkmark\ Q_{\leq k}(L) = \sum_{j=0}^{k} Q_j(L)$$

<u>Theorem</u>: $Q_{\leq k}(L) = \Theta(nk)$

Proof: Fix a parameter $0 \le p \le 1$

Choose each line of L with prob. p.

R: set of chosen lines.

$E[|R|] = pn$

$A_0(R)$: level 0 of R.

$V_R(L)$: set of vertices of $A(L)$ whose level 0 R.

$Q_R(L) = |V_R(L)|$

$$E[Q_0(R)] = \sum_{v \in A(L)} \Pr[v \in V_0(R)]$$

$$= \sum_{j=0}^{n-1} \sum_{v \in V_j(L)} \Pr[v \in V_0(R)]$$

$$= \sum_{j=0}^{n-1} \sum_{v \in V_j(L)} p^2 (1-p)^j$$

$$\geq \sum_{j=0}^{k} p^2 (1-p)^j Q_j(L)$$

$$\geq \sum_{j=0}^{k} p^2 (1-p)^k Q_j(L)$$

$$= p^2 (1-p)^k Q_{\leq k}(L)$$

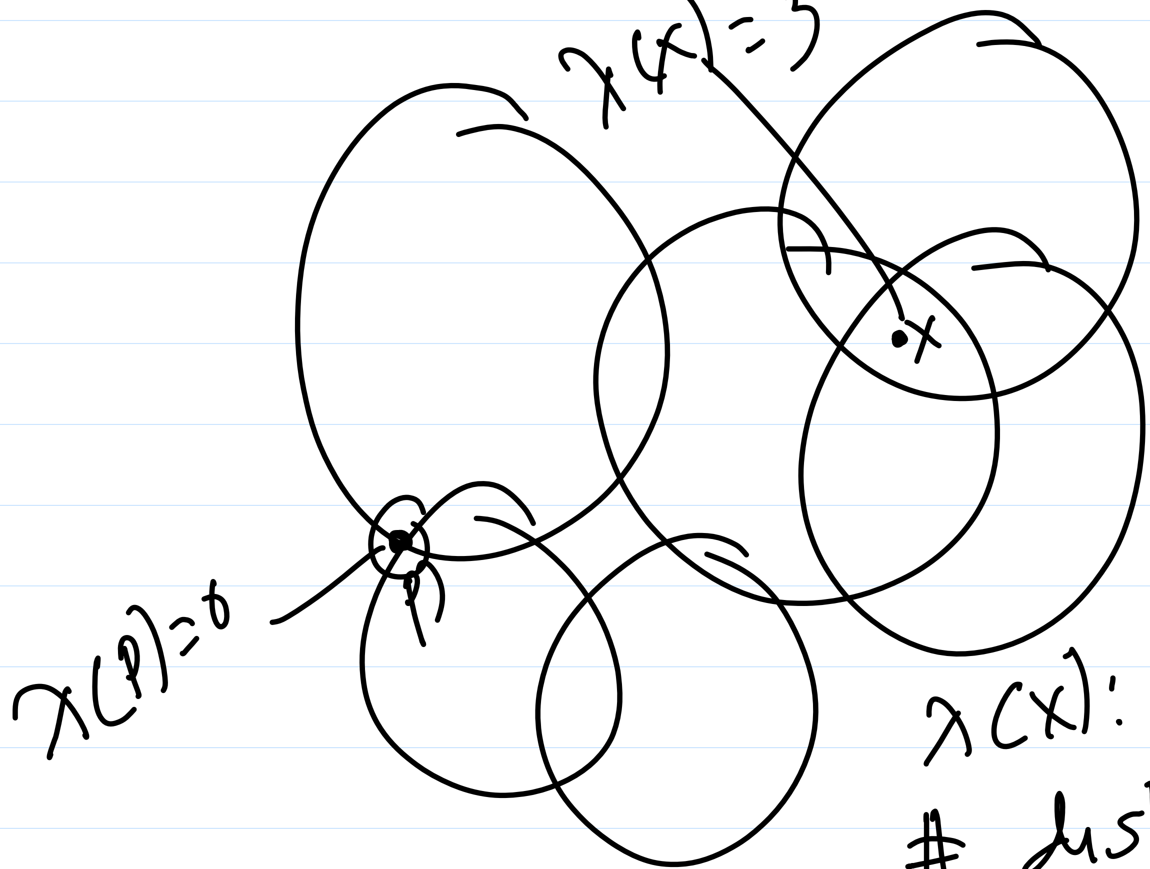
$$\boxed{Q_{\leq k}(L) \leq E[Q_0(R)] / p^2 (1-p)^k}$$

Set $p = 1/k$

$$\underline{Q_{\leq k}(L)} \leq \frac{k^2}{\left(1-\frac{1}{k}\right)^k} \cdot \frac{n}{k}$$

$$\leq e \cdot nk = O(nk)$$

$$\boxed{Q_{\leq k}(n) \leq k^2 \cdot e \, Q_0(n/k)}$$



$D = \{D_1, \dots, D_n\}$: n disks

$x \in \mathbb{R}^2$

$\lambda(x)$: level of x

disks that contain x in their interior.

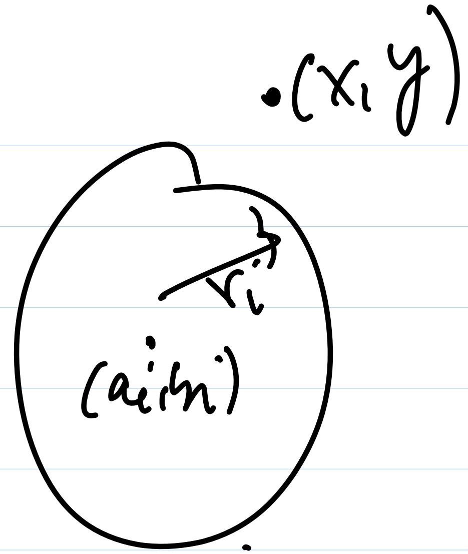
$A_0(D)$: Boundary of $\bigcup_{i=1}^{n} D_i$

$A_k(D)$: k-level

$Q_k(D)$, $Q_{\leq k}(D)$

$\boxed{\underline{\text{Lemma 1}}\text{: } Q_0(D) = O(n)}$

<u>Lemma 2</u>: $Q_{\leq k}(D) = O(nk)$



$$D = \{D_1, \dots, D_n\}$$

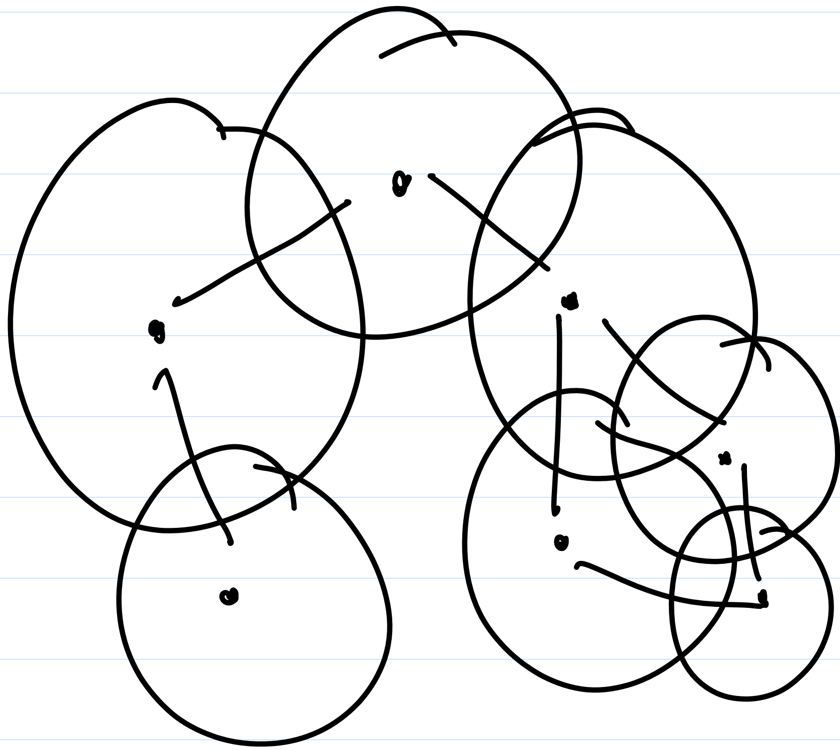
$$D_i = (a_i, b_i),\ r_i$$

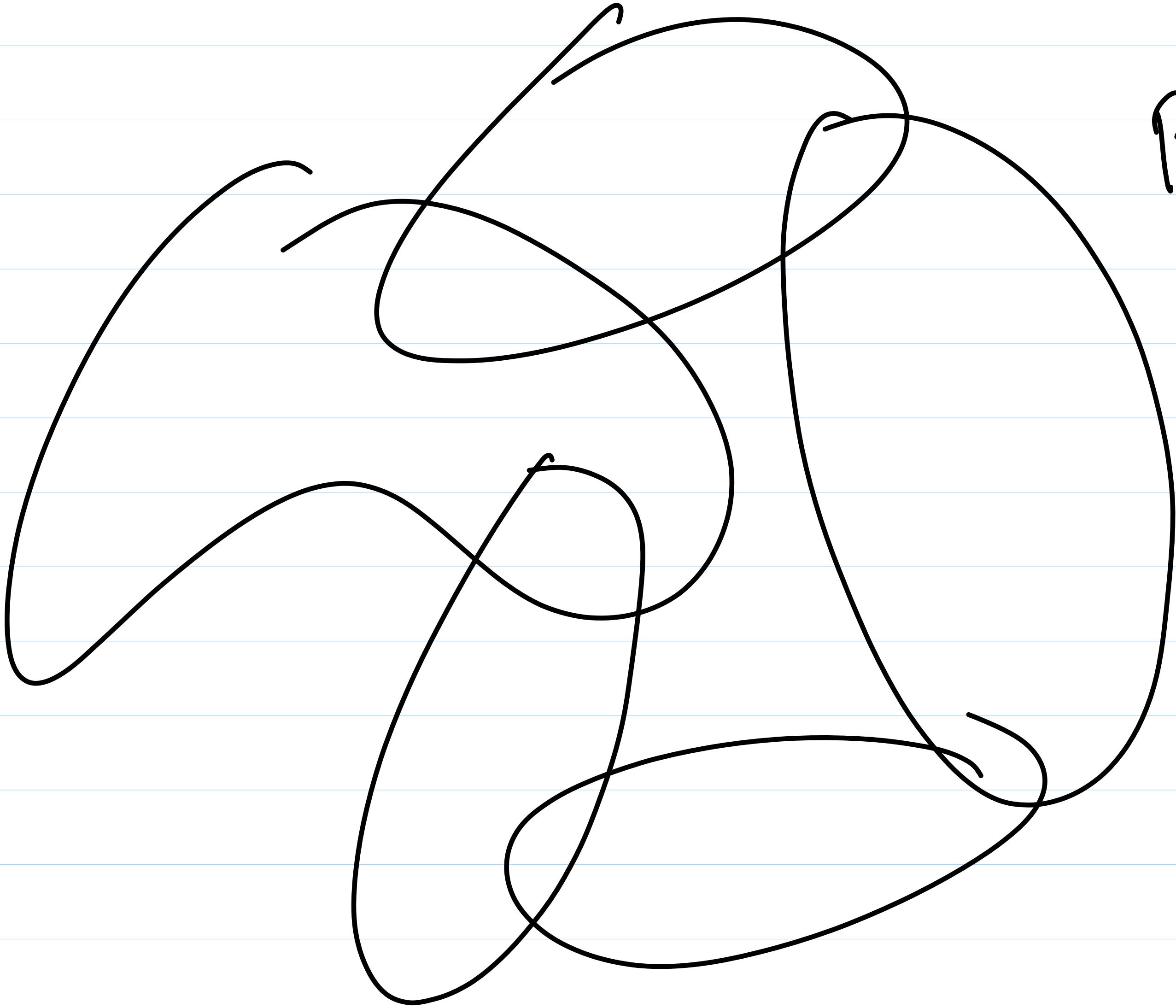
$$(x-a_i)^2 + (y-b_i)^2 \geq r_i^2 \quad \forall i \quad \text{if } (x,y) \notin \bigcup_{i=1}^{n} D_i$$

$$\underbrace{x^2+y^2}_{=z} - 2a_i x - 2b_i y + a_i^2 + b_i^2 \geq r_i^2$$

$$h_1:\ z \geq 2a_i x + 2b_i y + r_i^2 - a_i^2 - b_i^2 \quad \forall i$$

$\bigcap_{i=1}^{n} h_i$ has $O(n)$ vertices





Pseudo disks