Lecture 19: Delaunay Triangulation

Delaunay disks: are defined as the interior of the circumcircle of a Delaunay Δ.

Delaunay disks, say \( D \cap S = \emptyset \) (without considering the points on the boundary)
The point set $S$ does not have 4 co-circular points (general position).

A simple algorithm for Delaunay $\Delta$-tim

We start with some arbitrary $\Delta$-tim. An arbitrary $\Delta$-tim will contain $2n - 2 - k$ $\Delta$s where $k$ is the # points on the convex hull of $S$. 
We examine a pair $\Delta'$ that share an edge

Does the $D(T_i)$ contain the point of $T_2$ (on the opposite side of the common edge )

$T_4$ and $T_5$ are legal (diagonal is legal)

and now check if there exists another pair of $\Delta'$ that are not legal

until all diagonals are legal

If $C$ was on the circle

$\xi < \alpha < \xi'$

$\Rightarrow$ the smallest angle of the $\Delta'$
The smallest angle of the triangulation defined by the "legal" edge is larger than the smallest angle of the $\Delta\text{trim}$ defined by the "illegal edge".

When we continue flipping the diagonals, the max angle of the triangulation increases. Delaunay $\Delta\text{trim}$ maximises the max angle among all $\Delta\text{trim}$. 
The convergence takes \( O(n^2) \) edge flips.

(Using appropriate data structures, you can obtain \( O(n^2) \) runtime.)

If you are given an arbitrary \( \Delta \)-tin, how quickly can you verify that it is a Delaunay \( \Delta \)-tin?

\( O(n) \) as check if any point lies within the deck.

\( \Theta(n^2) \) procedure.
If we can verify for every edge that the two \( \Delta \)'s sharing the edge are legal, then it is a D.T.

Local Delaunay property

\[ \Rightarrow \] Global property

Randomized incremental construction

Given the set of points \( S = \{ P_1, P_2, \ldots, P_n \} \),

first generate a random permutation of \( S \), say \( Q = \{ q_1, q_2, \ldots, q_n \} \)

Enclose the points in some large \( \Delta \) and then
→ Expected degree of $g_i$
    \[ = O(1) \]

→ Main being the $\Delta$ in current DT.
Claim: The randomized incremental construction for Delaunay triangulation takes expected $O(n \log n)$ time for any arbitrary set of $n$ points.