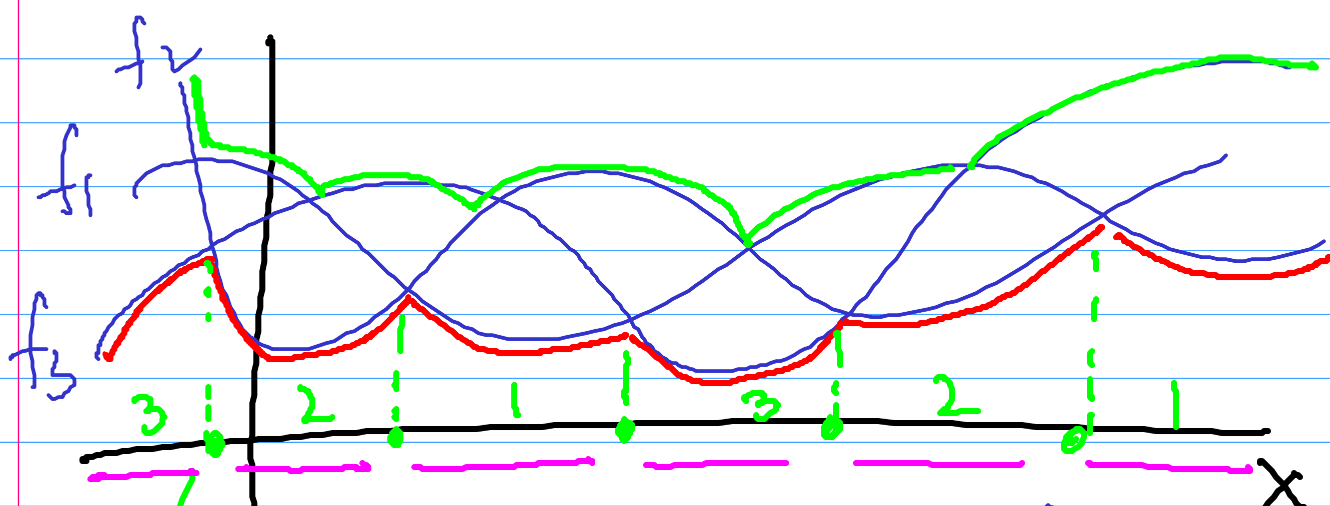
Lecture 18: Voronoi Diagram & Delaunay Triangulation

$S = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$

$Vor(p_i) = \{x \mid \|x p_i\| \leq \|x p_j\| \; \forall j \in S\}$

Lower Envelope

$F = \{f_1, \dots, f_n\} \quad f_i : \mathbb{R}^2 \to \mathbb{R}$

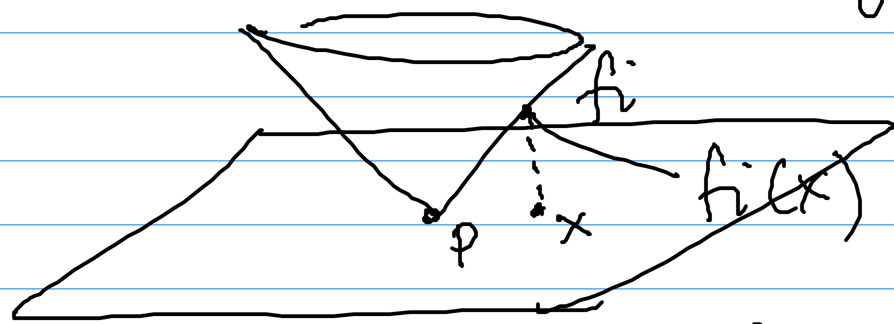


$L_F(x) = \min_{1 \leq i \leq n} f_i(x)$

Minimization diagram:
Partition of x-axis into intervals s.t. the same f_i appears on L_F within each interval.

$f_i(x) = \| x \, \overset{(a_i, b_i)}{\overset{\shortparallel}{p_i}} \|$ Distance from x to p_i

$$f_i(x, y) = \sqrt{(x - a_i)^2 + (y - b_i)^2}$$



$F = \{f_1, \ldots, f_n\}$

$L_F(x, y) = \min_i f_i(x)$ Distance from x to the nearest point in S

Minimization diagram of F

Within each region same f_i appears on L_F. $\Longrightarrow$ $Vor(p_i)$

$\Longleftrightarrow$ Voronoi Diagram of S.

Paraboloid $g_i(x,y) = f_i^2(x,y)$

$$= (x-a_i)^2 + (y-b_i^2)$$

$$= x^2+y^2-2a_ix-2b_iy + a_i^2+b_i^2$$

plane

$$h_i(x,y) = g_i(x,y) - x^2 - y^2$$

$$= -2a_ix - 2b_iy + a_i^2 + b_i^2$$

$$H = \{h_1, \dots, h_n\}$$

<u>Lemma</u>

$\mathrm{Vor}(S)$ = Minimization dia. of F

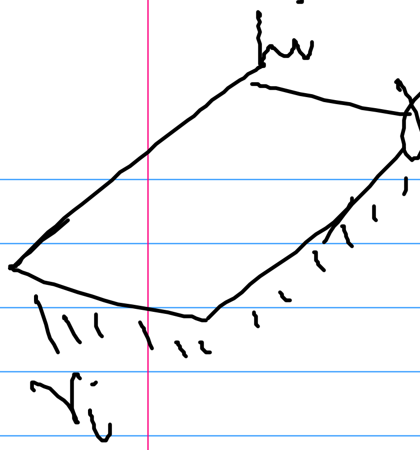
= Minimization dia of H

<u>Proof</u>:

$$f_i(x,y) \le f_j(x,y) \iff f_i^2(x,y) \le f_j^2(x,y)$$

$$\Updownarrow$$

$$h_i(x,y) \le h_j(x,y) \iff \underbrace{f_i^2(x,y) - x^2 - y^2}_{h_i(x,y)} \le \underbrace{f_j^2(x,y) - x^2 - y^2}_{h_j(x,y)}$$



$\gamma_i : \{(x,y,z) \mid z \le h_i(x,y)\}$

Lower Envelope of $H = \bigcap_{i=1}^{n} \gamma_i$ in $\mathbb{R}^2$

Computing Vor(S) $\Longrightarrow$ Computing intersection of n half spaces in $\mathbb{R}^3$

$\bar{h}_i(x,y) = 2a_i x + 2b_i y - a_i^2 - b_i^2$

Dualize $\bar{h}_i$ to a point h_i^* in $\mathbb{R}^3$

$h_i^* = (2a_i, 2b_i, a_i^2 + b_i^2)$

$H^* = \{h_i^* \mid 1 \le i \le n\}$

Dual of $\bigcap \gamma_i \Leftrightarrow \text{conv}(H^*)$

conv(H^*) can be computed in $O(n \log n)$ time.

$$h_i^{\#} = (a_i, b_i, a_i^2 + b_i^2)$$

$$H^{\#} = \{ h_i^{\#} \mid 1 \le i \le n \}$$

$$S = \{ (a_i, b_i) \mid 1 \le i \le n \}$$

$$C : (\alpha_i, \beta_i, \gamma_i)$$

$$(x - \alpha_i)^2 + (y - \beta_i)^2 = \gamma_i^2$$

$$x^2 + y^2 - 2\alpha_i x - 2\beta_i y + \alpha_i^2 + \beta_i^2 = \gamma_i^2$$

$$z = 2\alpha_i x + 2\beta_i y + \gamma_i^2 - \alpha_i^2 - \beta_i^2$$

$$(x, y) \longrightarrow (x, y, x^2 + y^2)$$