COL 702 July 27
Lecture 2

Shortest Path Algorithms — continued

Navigation from source A to dest B

→ mapped to a problem in graph

$G = (V, E)$ directed and weighted

$w: E \to \mathbb{R}$

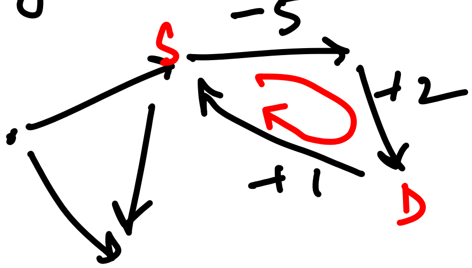
Dijkstra algorithm : Single source shortest path (SSSP)

$w: E \to \mathbb{R}^+$ (No -ve wts)

Bellman-Ford : SSSP

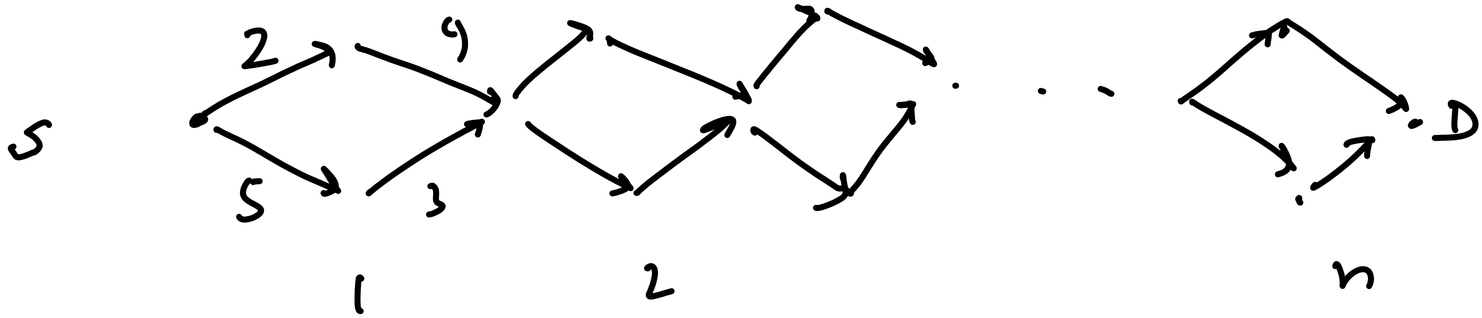
no -ve cycle

Directed Acyclic Graph (DAG)



Inspite of -ve cycles, can we demand a shortest path that has no cycles

Simple path : $v_{i_1} \to v_{i_2} \to v_{i_3} \to \cdots \to v_{i_k}$



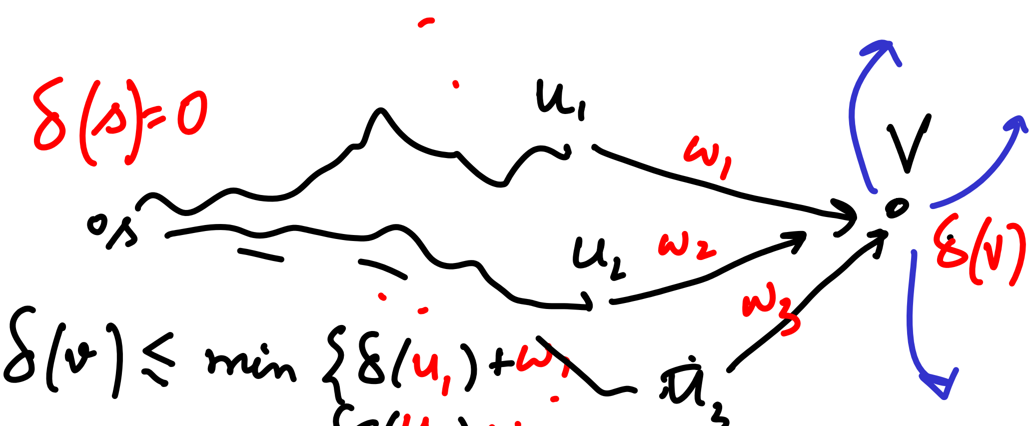
How many paths between S and D?

Brute force $\Rightarrow$ exponential-time

Common step between B F and D

<u>Shortcut step</u>

Maintain a good upper bound from source to all vertices $v \in V$ starting with ∞



$$\delta(v) \le \min\{\delta(u_1)+w_1,\ \delta(u_2)+w_2,\ \delta(u_3)+w_3\}$$

$\delta(v)$: upper bound

Gradually these algorithms refine the estimates iteration

<u>Induction invariant</u>: There is a path from s to v with distance $\le \delta(v)$

Shortcut step (Relaxation step)

Relax edge (u, v)

if $\delta(v) > \delta(u) + \omega(u,v)$

then $\delta(v) \leftarrow \delta(u) + \omega(u,v)$

In BF, we have $|V|-1$ iteration where in each iteration, we relax all edge $(u,v) \in E$

(ordering doesn't matter)

Running time : $O(|V|.|E|) \sim O(n^3)$

$|V| = n \quad |E| = m$ $\qquad |V| = n$

Correctness : — No path can have more than $|V| - 1$ edges

Claim : All vertices v whose shortest path from the source s consists of k links will have $\delta(v) = D(v)$ with k iterations

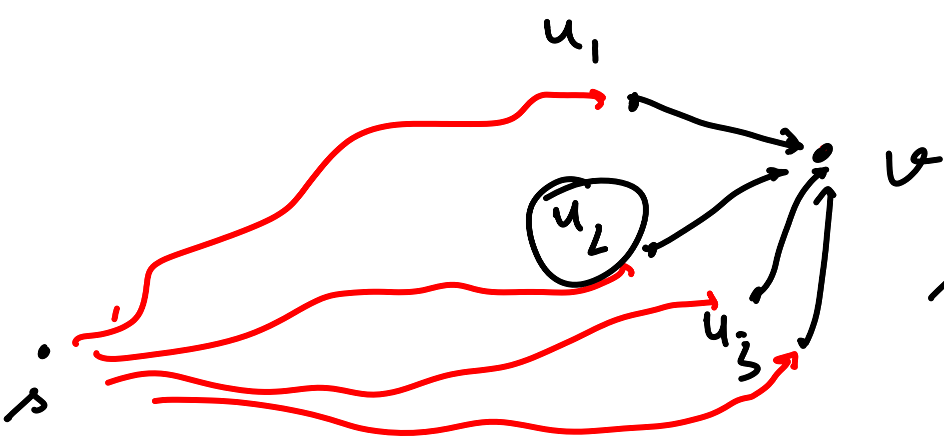
shortest path length

Proof by induction (on k)

Basis: $k = 0$: by initialization it is correct

Suppose true for $[0, 1, 2, \ldots i-1]$ → i

$[i-1]$



(subpaths of all shortest paths are shortest)

Shortest path to v

$\underbrace{s \to x_1 \to x_2 \to x_3 \cdot u_2}_{\text{shortest path to } u_2 \text{ with } i-1 \text{ edges}} \to v$

⇒ Shortest path to u_2 must have been computed with $i-1$ iteration by I.H.

In the i^{th} iteration, when we relax edge (u_2, v) $\delta(v) = D(v)$