Constructing Graph Spanners

The girth of a graph (shortest cycle) is critical for bounds on the "stretch" of a spanner.

In particular \( \frac{\delta_{G_s}(x,y)}{\delta_G(x,y)} \geq g-1 \)

where \( g \) is the girth.

A complete bipartite graph has girth \( g=9 \)

\( \Rightarrow \) any spanner has stretch \( \geq 3 \)

Observe that the denser the graph, the smaller is the girth.

There are favorable trade-offs between 
- \( \# \) edges 
- and girth of a graph.

If a graph \( \geq m \) edges then girth \( \leq f(m,n) \)

for some function \( f \) of \( m \)
There are graphs with \( m = kn^{1 + \frac{1}{k}} \) edges that have girth \( \leq 2k \) for any integer \( k \).

E.g. \( k = 2 \) \( n^{1 + \frac{1}{2}} = n^{3/2} \) has a cycle \( \leq 4 \) \( \text{stretch} \ 3 \).

\( k = 3 \) \( n^{4/3} \) edges girth \( \leq 6 \) \( \text{stretch} \ 5 \).

\( k = \log n \) \( m = \log n \cdot n \cdot n \cdot \log n \).

\( \text{girth} - 2 \log n \) \( \text{stretch} \sim \log n \).

\( \text{stretch} : \text{girth} - 1 \)
**Phase 1 :** 0) Choose \(\sqrt{n}\) vertices randomly, say \(R \subset V\)

For \(v \in V - R\), we find the nearest sampled neighbor \(n_v\) in \(R\) and form clusters based on \(n_v\).

\(n_v\) is undefined if none of the neighbors \(N(v)\) are sampled.

So there are \(\sqrt{n}\) clusters — one for each of the sampled vertices.

1) For every vertex \(v \in V - R\), choose all edges \((v, x)\) to include in \(E_s\) s.t. \(\omega(v, x) \leq \omega(v, n_v)\) if no neighbor \(n_v\) is sampled, add all edges out of \(v\).
Phase 2

For any vertex $v$, consider the "inter-cluster" edges $C(v)$ in the cluster of $v$.

Among the inter-cluster edges, choose the least weighted for every cluster and add to $E_s$.

Total # of edges added to $E_s$ in phase 2:

$\sum_{i=1}^{\sqrt{n}} E_s = n \times \sqrt{n}$ (number of clusters)

Expected # of edges added to $E_s$ in phase 1:

$\sqrt{n}$ per vertex using linearity of $E|x^n$.
Sketch of the graph $G_s : (V,E_s)$

Consider an edge $(x, y) \in E - E_s$

**Case I**

$x, y$ belong to the same cluster

$C(y) = C(x)$

$\eta_x = \eta_y = z$

$\omega(x, z)$

**Case II**

$C(x) \neq C(y)$

\[ \omega(x, t) \leq \omega(x, y) \leq \omega(z, y) \leq \omega(x, y) \leq \omega(x, y) \]
What is the running line of Phases I and II?