Job scheduling problem

\[ J_1, J_2, J_3, \ldots, J_n \]

Time required \[ \Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n \]

Deadline \[ d_1, d_2, d_3, \ldots, d_n \]

Penalty \[ p_1, p_2, p_3, \ldots, p_n \]

\( \Delta_i, d_i \) are integral

Objective: Minimize penalty

Example: \( \Delta_i = 1, d_i = 1 \)

Then almost one job can be completed within the deadline.

More general situation \( \Delta_i = 1 \) but \( d_i \)'s are distinct
Ex. 2

<table>
<thead>
<tr>
<th></th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$J_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_i$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$p_i$</td>
<td>5</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

**Obs:** 1. A job $J_i$ with deadline $d_i$ can be scheduled in any of the time slots $1, 2, \ldots, d_i$ if we want to avoid penalty.

2. Minimize the penalty if the jobs that miss their deadlines are the same as minimizing penalty of jobs that are scheduled within deadlines.

If we apply a greedy solution to this problem, then we should pick them in decreasing order of penalty and check feasibility using some efficient procedure.
A feasible set of jobs \( A \subset J \) can be scheduled within their respective deadlines.

**Question**: Given a set of jobs \( A \), is it feasible?

**How assignment**: complete with proof.

The set of feasible jobs satisfies the subset system property.

Is this subset system a matroid?

1. Exchange property
2. Rank property

Do these hold?

Given subsets \( A, B \) of jobs s.t.

\( A, B \) are feasible \([A \cap H_i \subseteq B]\), can we choose some \( J_k \subseteq B - \{x\} \cup \{J_x \} \) to make \( A \cup J_k \) feasible?
A, B ∈ I: family of subsets that are feasible

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>i</th>
<th>K</th>
<th>K+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>J₁</td>
<td>J₂</td>
<td>J₃</td>
<td></td>
<td></td>
<td>Jᵢ</td>
<td>J_k</td>
</tr>
<tr>
<td>B</td>
<td>J'_₁</td>
<td>J'_₂</td>
<td>J'_₃</td>
<td></td>
<td></td>
<td>J'_k</td>
<td>J'_k+1</td>
</tr>
</tbody>
</table>

Case 1: J'ₖ₊₁ ∉ A, then A ∪ J'ₖ₊₁ is feasible

Case 2: J'ₖ₊₁ ∈ A i.e. J'ₖ₊₁ = Jᵢ ∈ A

By induction we can prove exchange property

What can we hope for greedy even when it is not a matroid?
Return an optimal solution of max $f(x_1, x_2, x_3, x_4)$, where $x_i = 1, 2, \ldots, b_i$, $i = 1, 2, 3, 4$. Choose the best set of $x_i$ when we pick $x_i$.

For example, the table shows the profits and weights of each item.

<table>
<thead>
<tr>
<th>item</th>
<th>weight</th>
<th>profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Choose $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$. Pick the highest in descending order.

Then the knapsack problem becomes

\[ \text{maximize } f(x_1, x_2, x_3, x_4) = 6 + 11 + 17 + 5 = 49. \]
What can we claim about $G$ (vis-a-vis the optimal profile $O$)?

$$\frac{G}{O} \geq 10\%, 20\% \ldots 50\%, 95\%?$$

$$\lambda + \beta \geq 0 \implies \text{either } \lambda \text{ or } \beta \geq \frac{O}{2}$$

$$\max \{ \lambda, \beta \} \geq \frac{O}{2}$$

$$\implies 50\% \text{ optimum}$$

Matching can be solved in polynomial time.

**Exercise**: Run greedy that produces a solution $G$.

Compare with optimal $O$.

**Hint**: 

![Diagram showing greedy and optimal solutions]