COL 702 Lecture 15 Sept 17

Matroid thm : $M = (S, \mathcal{I})$ feasible subsets

ground set

. The following are equivalent

(1) Generic greedy works correctly for all wt functions

(2) Exchange property holds

(3) Rank property holds

$$① \Rightarrow ② \Rightarrow ③ \Rightarrow ①$$

$① \Rightarrow ②$ If greedy works then exchange property holds

$S_1, S_2 : |S_1| < |S_2| \quad S_1, S_2 \in \mathcal{I}$

Then $\exists\ x \in S_2 - S_1$ s.t. $S_1 \cup \{x\} \in \mathcal{I}$

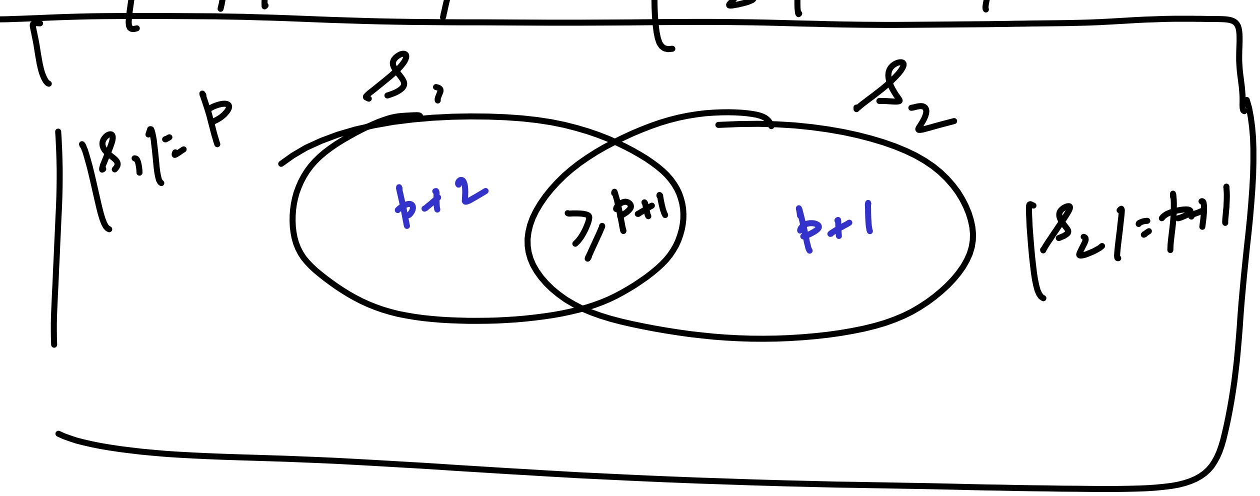
Wlog $|S_2| = |S_1| + 1$

Proof by contradiction. Suppose exchange property does not hold for some pair S_1, S_2

There is no element $x \in S_2 - S_1$ s.t $S_1 \cup \{x\} \in \mathcal{I}$

We want to show that generic greedy fails for some wt function

$|S_1| = p \qquad |S_2| = p+1$



$$w(e) = \begin{cases} p+2 & \text{if } e \in S_1 \\ p+1 & \text{if } e \in S_2 - S_1 \\ 0 & \text{otherwise} \end{cases}$$

Greedy soln : $(p+2) \cdot p$

(No further element from $S_2 - S_1$ can added by greedy according to our assumption that exchange property doesn't hold

The profit yield from picking all elements of $S_2 \geq (p+1)^2$

Since $(p+1)^2 > p^2 + 2p$ Contradiction that greedy works

② $\Rightarrow$ ③ is obvious

③ $\Rightarrow$ ① Rank property : For any subset $A \subseteq S$ that all maximal subsets of A have the same cardinality

Suppose rank property holds but greedy fails

Suppose greedy produces soln (in decreasing order of wts)

e_1	e_2	e_3	$\dots$	e_j	$\dots$	e_k
e_1'	e_2'	e_3'	$\dots$	e_j'	$\dots$	e_l'

better soln also in decreasing order of wts

$$\sum w(e_i) < \sum w(e_i')$$

Because of the rank property $k = l$

Let j be the smallest index $w(e'_j) > w(e_j)$

$$A = \{x \mid w(x) \geq w(e'_j)\}$$

Note $e_1, e_2, \ldots e_{j-1}$ is maximal in A with cardinality $j-1$

whereas $\{e'_1, e'_2, \ldots e'_j\}$ has cardinality at least j

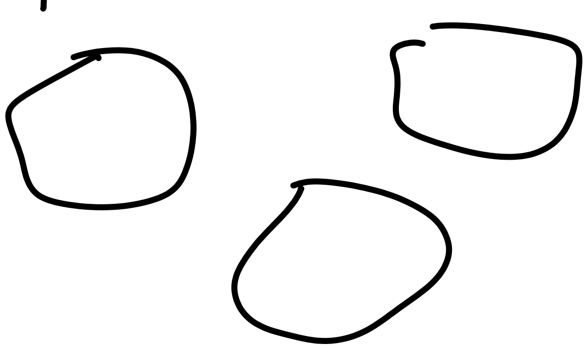
contradiction

Applications of the Matroid thm

Maximal Spanning Tree/Forest

Rank property holds?

Given a forest what is the rank

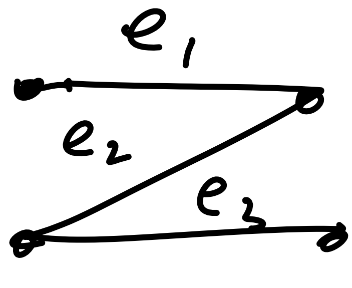


A forest with K components has rank $n - K$

For a connected graph with n vertices, the tree has $n-1$ edges

1 Knapsack : Construct examples where rank property fails

2 Matching

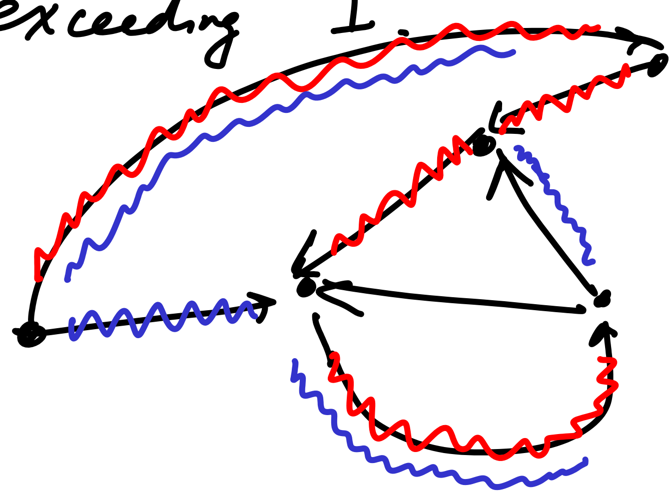


$\{e_2\}$

$\{e_1, e_3\}$

Rank property fails

3. Half-matching : Given a directed weighted graph $G = (V, E)$, we want to pick a maximum weighted set of (directed) edges so that no vertex has indegree exceeding 1.



Exchange property : Consider two independent subsets S_1 S_2 $|S_1| = p$ $|S_2| = p+1$

Can we move some edge from S_2 to S_1 s.t. indegree of all vertices in $S_1 \leq 1$

Observation : Let V_1, V_2 be vertices in S_1, S_2 resp with indegree = 1. Since $|V_2| > |V_1|$, one such edge can be shifted to S_1 without increasing indegree to 2.

What about minimizing the sum of weights, say minimum Spanning Tree?

Can we reduce this problem to an instance of the matroid.

Define a new wt function

$w'(e_i) = W - w(e_i)$ where W is the maximum wt

Now run the generic greedy and suppose it picks up wts

$w_1', w_2', w_3' \ldots w_k'$ sol

$\sum w_i'$ is maxm among all possible independent subset

$$\Rightarrow \sum (W - w_i') = kW - \sum w_i \text{ is maxm}$$

$\Rightarrow \sum w_i$ is minimum

because kW is a fixed quantity for the rank property