Matroid then: \( M = (S, I) \)

The following are equivalent:

1. Generic greedy works correctly for all wt functions.
2. Exchange property holds.
3. Rank property holds.

\( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \)

\( 1 \Rightarrow 2 \) If greedy works then exchange property holds:

\( \beta_1, \beta_2 : |\beta_1| < |\beta_2|, \beta_1, \beta_2 \in I \)

Then \( I \ni x \in \beta_2 - \beta_1 \), s.t. \( x, \cup \{2\} \in I \)

Wlog \( |\beta_2| = |\beta_1| + 1 \)

Proof by contradiction. Suppose exchange property does not hold for some pair \( \beta_1, \beta_2 \).
There is no element \( x \in S_2 - S_1 \), s.t. \( S, U \{ x \} \notin \mathcal{I} \)

We want to show that generic greedy fails for some \( \mathcal{I} \) function.

\[
|S_1| = p \quad |S_2| = p + 1
\]

\[
|S_1| = p \quad S_1 \quad S_2 \quad |S_2| = p + 1
\]

\[
\omega(e) = \begin{cases} 
  p+2 & \text{if } e \in S_1 \\
  p+1 & \text{if } e \in S_2 - S_1 \\
  0 & \text{otherwise}
\end{cases}
\]

Greedy soln: \((p+2) \cdot p\)

(No further element from \( S_2 - S_1 \) can be added by greedy according to our assumption that the exchange property does not hold.)
The profit yield for picking all elements of \( S_2 \geq (p+1)^2 \)

Since \((p+1)^2 > p^2 + 2p\) contradiction that greedy works.

\(\Theta \Rightarrow \Theta \) in obvious.

\(\Theta \Rightarrow \Theta \) Rank property: For any subset \( A \subseteq S \) that all maximal subsets of \( A \) have the same cardinality.

Suppose rank property holds but greedy fails.

Suppose greedy produces solution in decreasing order.

\[
\begin{array}{cccccc}
  e_1 & e_2 & e_3 & \cdots & e_j & \cdots & e_k \\
  e_1' & e_2' & e_3' & \cdots & e_j' & \cdots & e_k'
\end{array}
\]

be the set also in decreasing order.

\[ \sum w(e_i) < \sum w(e_i') \]

Because of the rank property \( k = \ell \)
Let $j$ be the smallest index such that $w(e_j) > w(e_j')$.

$$A = \{ x \mid w(x) > w(e_j) \}$$

Note: $e_1, e_2, \ldots, e_{j-1}$ is maximal in $A$ with cardinality $j-1$.

where \( \{ e_1, e_2, \ldots, e_j \} \) has cardinality at least 1.

Applications of the Matroid Theorem

Maximal Spanning Tree/Forest

Rank property holds?

Given a forest what is the rank?

A forest with $K$ components has rank $n-K$.

For a connected graph with $n$ vertices, the tree has $n-1$ edges.
1. Knapsack: Construct example where rank property fails

2. Matching

<table>
<thead>
<tr>
<th>$e_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_2$</td>
</tr>
<tr>
<td>$e_3$</td>
</tr>
</tbody>
</table>

$\{e_2\}$

$\{e_1, e_3\}$

Rank property fails

3. Half-matching: Given a directed weighted graph $G = (V, E)$, we want to pick a set $F$ (directed) edges so that no vertex has indegree exceeding 1.

Diagram of a directed graph with vertices and directed edges.
Exchange property: Consider two independent subsets $S_1, S_2$ with $|S_1| = \rho, |S_2| = \rho + 1$.

Can we move some edge from $S_2$ to $S_1$, s.t. indegree of all vertices in $S_1 \leq 1$?

Observation: Let $V_1, V_2$ be vertices in $S_1, S_2$, respectively, with indegree $= 1$. Since $|V_2| > |V_1|$, one such edge can be shifted to $S_1$, without increasing indegree 2.

What about minimizing the sum of weights, say minimum spanning tree?

Can we reduce this problem to an instance of the matroid?

Define a new cost function $w'(e) = w - \omega(e)$ where $\omega$ is the maximum wt.

Now run the generic greedy algorithm and suppose it picks up sets.
\[ \omega_1, \omega_2, \omega_3, \ldots, \omega_k \text{ sol} \]
\[ \leq \omega_i \text{ is maxm among all possible independent subcl} \]
\[ \leq (W - \omega_i) = kW - \sum \omega_i \text{ is maxm} \]
\[ \leq \omega_i \text{ is minimum} \]

Because \( kW \) is a fixed quantity, for the rank properly