What does the following DFA accept

\[ M \]

Difficult to characterize a language by looking at the DFA, but once we have a conjecture, we must prove it rigorously.

For the above machine

\[ L(M) = \{ \emptyset, 11, 00, 011, 110 \ldots \} \]

Clearly \( L(M) \) is infinite.

Do we have a nice succinct description?
Claim: \[ L(M) = \{ w | \text{\text{bin}}(w) \text{ is divisible by } 3 \} \]

\text{bin}(w) \text{ is the value of a string } w \text{ interpreted as a binary number}

How do we prove it?

Induction on the length of the string

\( \forall n \geq 1, \> w \in \Sigma^n \text{ is accepted by } M \text{ iff } \text{bin}(w) \mod 3 = 0 \)

Is the above easy to prove?

Base cases: 0, 1; strings of length one.

(i) 0 is accepted since \( 0 \) is a final state
(ii) 1 is not accepted as \( 0 \) is not a final state

I.H. Suppose the above is true for all strings of length \( n \), then we must prove it for all strings of length \( n+1 \)
Let \( |w'| = \eta + 1 \) where
\[
\begin{align*}
w' & \rightarrow w_0 \quad [w_1 = \eta] \\
\end{align*}
\]
So \( w \) satisfies I. H.

Case A: \( \omega \in L(M) \) i.e. \( \text{bin}(w) \mod 3 = 0 \)

So \( \text{bin}(w') = 2 \times \text{bin}(w) + 0 \)

\( \delta^*(q_0, w_0) = \delta(\delta^*(q_0, w), 0) = \delta(q_0, \eta) = q_0 \)

i.e. \( \omega_0 \) in accepted and \( \text{bin}(w_0) \mod 3 = 0 \)

So it is fine

\( \delta^*(q_0, w_1) = \delta(\delta^*(q_0, w), 1) = \delta(q_0, \eta) = q_0 \)

\( \text{bin}(w_1) = [\text{bin}(w) \times 2 + 1] \mod 3 = 1 \)

Since \( q_0 \) is not accepting it is fine

Case B: \( \omega \notin L(M) \), i.e. \( \text{bin}(w) \mod 3 \neq 0 \)

What can we say about \( \omega_0 \) or \( \omega_1 \)?

From DFA, if \( \text{bin}(w) \mod 3 = 1 \), then \( \omega_1 \) is accepted but if \( \text{bin}(w) \mod 3 = 2 \), then \( \omega_1 \) is not accepted.
So to complete the inductive proof we need more information about \( w \), i.e. \( \text{bin}(w) \mod 3 = 1 \) or \( 2 \)?

So the inductive assertion must have more than just the property \( f \) states, (which is the accepting state)

\[ \text{Attempt 2} \quad \forall n > 0, |w| = n \]

Claim

(i) \( \delta^*(q_0, w) = q_0 \) \( \iff \text{bin}(w) \mod 3 = 0 \)

(ii) \( \delta^*(q_0, w) = q_1 \) \( \iff \text{bin}(w) \mod 3 = 1 \)

(iii) \( \delta^*(q_0, w) = q_2 \) \( \iff \text{bin}(w) \mod 3 = 2 \)

[Or more compactly \( \delta^*(q_0, w) = q_i \) \( \iff \text{bin}(w) \mod 3 = i \) \( i = 0, 1, 2 \)]
Proof. Case \( \omega_1 = 1 \) \( \omega_0 = 0 \).

Check that \( \delta(9_0,0) = 9_0 \), \( \delta(9_0,1) = 9_1 \).

Since no string of length 1 reaches \( 9_2 \), it is vacuously true.

I.H. Suppose the assertion is true for all strings \( w, |w| = n \).

\[ w' = w \cdot 0 \text{ or } w \cdot 1 \]

Case A

Case B

\[ \text{Check: } \bin(w') = \bin(w) \times 2 \]

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<thead>
<tr>
<th>( \bin(w) \mod 3 = 0 )</th>
<th>( \bin(w) \mod 3 = 1 )</th>
<th>( \bin(w) \mod 3 = 2 )</th>
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\( \delta(9_0,0) = 9_0 \) \( \bin(w) \mod 3 = 0 \)

and \( \bin(w) \mod 3 = 2 \)

\( \bin(w) \mod 3 = 2 \)

\( \delta(9_1,0) = 9_2 \) \( \bin(w) \mod 3 = 3 \)

and \( \bin(w) \mod 3 = 2 \mod 3 \)

\( \bin(w) \mod 3 = 2 \times 2 \mod 3 = 1 \)
Case B

Similarly you complete Case B

\[ w' = w_1 \]

From these 2, we can conclude that

\[ 8^*(q_0, w) = q_i \text{ iff } \text{bin}(w) \mod 3 = i \]

(Otherwise it would have shown up in one of the above cases)

The claim implies that

\[ L(M) = \{ w | \text{bin}(w) \mod 3 = 0 \} \]

since

\[ q_0 \text{ is the only accepting state} \]