Finite alphabet (set of symbols) \( \Sigma \)

Strings: a sequence of symbols from \( \Sigma \) of finite length

\( S \in \Sigma^* \quad \epsilon : \) string of length 0

Language \( L \subseteq \Sigma^* \)

\( \chi_L (w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases} \)

Characteristic function for membership problem

\( \# \text{ languages} \rightarrow \Sigma^* \)

\( \# \text{ programs} \sim \Sigma^* \)

Classes of languages

Diagram: Overlapping sets
Hierarchy of languages using increasing resources for its recognition

"Chomsky" hierarchy

\[ A \xrightarrow{f} B \]

\[ \text{domain} \]
\[ \text{co domain} \]

\[ f \text{ is } 1:1 \quad \text{(one-to-one)} \]
\[ f \text{ is } 1:1 \text{ and onto: bijection} \]

When sets are finite, we can compare their cardinalities by the number of elements in sets.

How do we compare cardinalities of infinite sets?

When we can define a bijection between two infinite sets, then we conclude that they have the same cardinality.
\[ f_E(i) \mapsto 2i \]

\[ \Sigma^* \rightarrow \mathbb{Z} \]

strings of length 1, 2, 3, \ldots

- ordering

\[ 0 \quad 0 . \]

1.

\[ \mathbb{Z} \times \mathbb{Z} = \{ (a, b) \mid a, b \in \mathbb{Z} \} \]

\[ (1, 3), (5, 20) \in \mathbb{Z} \times \mathbb{Z} \]

\[ \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \]
$\mathbb{Z}^+ \cup \mathbb{Z} \cup \mathbb{Z}^+$

- Countable: all these infinite sets that have a bijection with $\mathbb{Z}$
  - Countable union of countable sets is countable
  - Reals are not equinumerous with Integers • Cantor's diagonalization
There cannot be a bijection between a set $S$ and its power set $2^S$.

**Case 1:** $S$ is finite. Trivial

$$i < 2^i \quad \forall \, i > 1$$

**Proof by contradiction for infinite sets**

Assume a bijection exists.

\[ S_1 \leq S \]

\[ S_2 \]

\[ S_3 \]

\[ S_d \]

\[ g(i) \text{ in the bijection} \]

$$S_d = \{ x \mid x \notin g^{-1}(x) \}$$

**Question:** $d \in S_d$
\[ d \in S_d \quad \Rightarrow \quad \text{accordy f} \]
\[ \text{defn } d \notin S_d \quad \Rightarrow \quad \text{accordy f} \]

### Diagonalization Proof

\[
\begin{array}{c|c}
\text{Math Induction} & \text{Complete Induction} \\
\forall i \ P(i) \text{ is true} & \\
P(0) \land \forall i \ (P(i) \Rightarrow P(i+1)) & P(0) \land \\
& \forall i \left[ (\land_{j \leq i} P(j)) \Rightarrow P(i+1) \right]
\end{array}
\]