Pumping lemma for CFL

Let $L$ be a CFL. Then there exist an integer $n$, s.t. for all $s \in L$ and $|s| \geq n$, $s$ can be written as $s = u \cdot v \cdot w \cdot x \cdot y$ where $|v| + |x| \geq 1$, $|v \cdot w \cdot x| \leq n$ and $\forall i \geq 0 \quad u \cdot v^i \cdot w \cdot x^i \cdot y \in L$

Example: $L = \{a^i \cdot b^i \cdot c^i, \quad i \geq 1\}$

Suppose $L$ is a CFL $\Rightarrow$ P.L. can be applied to $u \cdot v \cdot w \cdot x \cdot y$.

$s = a \cdot a \cdots a \cdot b \cdot b \cdots b \cdot c \cdots c \cdots c$
Proof of PL for CFL

Assume that the CFL is in CNF.
All production rules $A \rightarrow BC \mid a$
$s \in L$. Consider a derivation tree for $s$.

In any binary tree with $2^t$ leaf nodes,
there must be a path of length $\geq t$.

$\Rightarrow$ If $t \geq |V|$, then some variable
must be repeated in the derivation along
the longest path.

Note that a path of length $t$ has $t+1$
vertices.
\[ |V| = k \]

The path has length \( \geq k \)
and so \# variables is \( \geq k+1 \)
\( \implies \) some variable in repeat.

\[ A_i \in V \]

\[ A_1 = A_{10} \]

\[ n = 2^k \]
\( \uparrow \) P.L. \( n \)
$L = \{ w w^R \mid w \in (0+1)^* \}$

$L = \{ a^i b^j a^i b^j, \ i,j \geq 1 \}$

$aa b a a b \in L \quad a b a a b \notin L$

Is $L$ CFL?

Consider $a^n b^n a^n b^n = u[vwx]y$

Exercise Show $L$ is not CFL using PL

Is CFL closed under union?

$L_1$ is a CFL, say $S_1$ is the start symbol

$L_2$ is a CFL, $S_2$ is the start symbol

Add the production $S \rightarrow S_1 \mid S_2$

Keep the variable names disjoint between $L_1$ and $L_2$
If \( L_1 \) is \( CFL \) and \( L_2 \) \( CFL \)

is \( L_1 \cap L_2 \) \( CFL \)?

Can we do a product construction of two PDAs, \( M_1, M_2 \)?

\( Q = Q_1 \times Q_2 \)

Can we simulate two stacks by one stack?

\[
L_1 = \{ a^i b^i c^i \mid i, j > 1 \}
\]

\[
L_2 = \{ a^i b^i c^i \mid i, j > 1 \}
\]

\[
L = L_1 \cap L_2 = \{ a^i b^i c^i \mid i > 1 \}
\]

\( L \) is not a \( CFL \), whereas \( L_1 \) and \( L_2 \) are \( CFL \)

CFLs are not closed under intersection.
Corollary: CFL are not closed under complementation.

$L \cap \overline{L_2} = \overline{L \cup \overline{L_2}}$

Special case: $L_1$: CFL; $L_2$: Regular.

$L = L_1 \cap \overline{L_2}$ is a CFL.

Use product construction: only one stack required.
\[ L = \{ w w^r \mid w \in (0+1)^* \} \]

Is \( L \) a CFL?

Suppose \( L \) is a CFL

Consider \( L_1 = \{ a^i b^j a^i b^j, \ i, j \geq 1 \} \)

\[ L_2 = a^+ b^+ a^+ b^+ \]

in regular

\[ L \cap L_2 = L_1 \]

Contradiction since \( L \), is not CFL

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**Decision problem on CFL**

1. Given a CFL is it

   (i) finite  (ii) infinite  (iii) \( \varnothing \)

Let \( K = |V| \), so \( n = 2^k \) is the constant of the P.L. for a given CFG in Chomsky Normal Form.
Claim: If $L \neq \emptyset$, then the shortest string in $L$, say $s$, such that $|s| < n$

Suppose not, then, let $s'$ be the shortest string in $L$ and $|s'| \geq n$. Then, from P.I. $s' = u v w x y$

such that $s'' = u v^0 w x^0 y \in L$ and $|s''| < s'$ and $|v| + |x| \geq 1$

Contradiction.

So it suffices to check if any string of length $\leq n-1$ is in $L$ to check emptiness.

Claim: If $L$ is infinite then there must exist a string $s \in L$ s.t. $n \leq |s| \leq 2n-1$
Suppose the shortest string in $L$ has length $\geq 2n$. (Since $L$ is infinite, it must have strings of length $\geq 2n$)

From P. L. $B = u v w x y$ s.t.

$$B_0 = u v^* w x^* y \in L$$

where $|B_1 - n| \geq |B_0| < |B_1|$ since

$$n \geq |v| + |x| \geq 1$$

This implies that $B$ is not the shortest string and the argument is valid for any $|B_1| \geq n$.

We want to show that there must exist a string in the range

$$\mu \stackrel{2n-1}{\rightarrow}$$

Suppose there is none, so apply the previous argument to $B$ which is the shortest string in $L$ whose length $\geq 2n$. 