Pumping Lemma for Regular Languages

For every regular language \( L \),

there exists a sufficiently large \( n \),

for all strings \( s \in L, |s| \geq n \)

the exist \( u, v, w = s \)

\( |uv| \leq n, |v| \geq 1 \)

s.t. for all \( i \geq 0 \) \( u \cdot v^i \cdot w \in L \)
$L' = \{ 0^i 1^i \mid i \geq 0 \}$

Is $L'$ regular?

Suppose $L'$ is regular $\Rightarrow$ we can apply PL to $L$

Then there exist some $n$ and let us choose $x = 0^n 1^n$ as the string

$w = \underbrace{000000}_{n} \overbrace{111111111111111}^{n}$

By PL $u v i w \in L$ for all $i$ and thereby produce string $x \in L'$ s.t. $\#Os > \#1s$

So the assumption that $L'$ is regular is false
\[ L_{\text{prime}} = \{ 0^k \mid k \text{ is prime} \} \]
\[ \{ 0, 0^3, 0^5, 0^7, \ldots \} \]

Suppose \( L_{\text{prime}} \) is regular (P.L. content \( n \))
Choose a sufficiently long string, say \( 0^p \)

\[
\begin{array}{ccccccc}
\text{nw} & 0 & 0 & 0 & 0 & 0 & \text{vw} \\
\end{array}
\]

<table>
<thead>
<tr>
<th>u</th>
<th>v</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>l</td>
<td>m</td>
</tr>
</tbody>
</table>

\[ |u| = k \quad |v| = l \quad |w| = m \quad k + l \leq n \]

From P.L., \( uv^n w \in L_p \) for all \( n \geq 0 \)

\[ \Rightarrow 0, k + il + m \in L_p \]

Choose \( i = k + m \)
\[ k + (k + m)l + m \in \rho \]
\[ (k + m)(1 + l) \in \rho \]
$L_c = \{ 0^k \mid k \text{ is not prime} \}$

Is $L_c$ regular?

Claim: If $L$ is regular, then $L \setminus (\varepsilon^* - L)$ is regular.

Choose a DFA for $L$ and invert the Final and non-Final states.

It follows $L_c$ is not regular.

Claim: If $L_1, L_2$ are regular then $L_1 \cap L_2$ is regular.

DFA: $M_1, M_2$

Construct a DFA in the following manner:

$M = M_1 \times M_2 : \{ Q_1 \times Q_2, \Sigma, (q_0^1, q_0^2), F, \delta \}$

$\delta : (q_1, q_2), a = (\delta_1(q_1, a), \delta_2(q_2, a))$

$q_1 \in Q_1, q_2 \in Q_2$
\[ L, \bigcap L_2 = \overline{L_1 \cup L_2} \] (De Morgan's law proof)

Suppose for each symbol \( a \in \Sigma \)
we define a substitution function
\[ f(a) \quad \text{regular expression} \]