Axiomatic Semantic Models
Let \( R, S \subseteq E \times E \) be binary relations on events. \( R; S \) denotes the relational composition of \( R \) and \( S \).

\( R^{-1} \) denotes the inverse of relation \( R \).

\( \text{dom}(R) \) and \( \text{range}(R) \) are domain and range of \( R \) respectively.

\( R^?, R^+, R^* \) are the reflexive, the transitive, the reflexive-transitive closures of \( R \) respectively.

\([A]\) denotes an identity relation on set \( A \) i.e.
\([A](x, y) \triangleq x = y \land x \in A\)

\( R_x \) denotes relation \( R \) on events on location \( x \) i.e.
\( R_x = \{(a, b) \mid a.\text{loc} = b.\text{loc} = x\} \)
We introduce modification order in execution ... 

Now execution graph is $G = \langle E, \text{po}, \text{rf}, \text{mo} \rangle$ where \text{mo} denotes modification order and $G.\text{mo}$ is the set of \text{mo} edges in $G$.

**Axiom**: \text{mo} is a total order on the same-location writes or updates in $G$.

For a pair of same-location writes or updates $a$ and $b$ in $G$, either $G.\text{mo}(a, b)$ or $G.\text{mo}(b, a)$ holds.

$$\forall a, b \in G.\text{WU}. \ a.\text{loc} = b.\text{loc} \implies G.\text{mo}(a, b) \oplus G.\text{mo}(b, a)$$

**Derived Relation**
From-read $\text{fr} \triangleq \text{rf}^{-1}; \text{mo} \setminus [E]$
Coherence

Restrictions on same-location events in an execution.

**Axioms**

- rf; po is irreflexive. (ConsRFPO)
- mo; po is irreflexive. (ConsWW)
- mo; rf; po is irreflexive. (ConsRW)
- rf⁻¹; mo; po is irreflexive. (ConsWR)
- rf⁻¹; mo; rf; po is irreflexive. (ConsRR)

\[ \text{COH} = \text{ConsRFPO} \land \text{ConsWW} \land \text{ConsRW} \land \text{ConsWR} \land \text{ConsRR} \]

**Restricted Patterns**

\[
\begin{align*}
R_x & \quad W_x \\
W_x & \quad W_x & \quad W_x \rightarrow R_x & \quad W_x \rightarrow W_x & \quad W_x \rightarrow W_x \rightarrow R_x
\end{align*}
\]
\( X := 1; \quad a := X; \quad // 2 \quad \| \quad X := 2; \quad b := X; \quad // 1 \)

Disallow \( a = 2, b = 1 \) outcome.
Atomicity

In an update event the read-write operations are atomic.

Axioms
- \( rf \) is irreflexive. (ConsIrRF)
- \( mo; rf \) is irreflexive. (ConsMORF)
- \( rf^{-1}; mo; mo \) is irreflexive. (ConsRFMOMO)

\( RMW = \text{ConsIrRF} \land \text{ConsMORF} \land \text{ConsRFMOMO} \)

Restricted Patterns
Initially $X = 0$;

$$a := \text{CAS}(X, 0, 1); \quad \parallel \quad b := \text{CAS}(X, 0, 1);$$

Both CASes cannot be successful in an execution.
Release-Acquire (RA) Model

Use *happen-before* relation: \( \text{hb} \triangleq (\text{po} \cup \text{rf})^+ \)

**Axioms**

- \( \text{hb} \) is irreflexive.  
  (ConsHB)
- \( \text{mo}; \text{hb} \) is irreflexive.  
  (ConsHBMO)
- \( \text{rf}^{-1}; \text{mo}; \text{hb} \) is irreflexive.  
  (ConsHBMORF)
- \( \text{rf}^{-1}; \text{mo}; \text{mo} \) is irreflexive.  
  (ConsRFMOMO)

**Example:**

\[
X = Y = 0; \\
X := 1; \\
Y := 1; \\
\begin{array}{l}
  a := Y; \quad // \ 1 \\
  \begin{array}{l}
    \text{if}(a == 1) \\
    b := X; \quad // \ 0
  \end{array}
\end{array}
\]

RA disallows \( a = 1 \land b = 0 \).
Axioms

- \((p_0 \cup r_0 \cup m_0 \cup f_r)\) is irreflexive. (SC)

Example: SC disallows \(a = 0 \land b = 0\).

\[
\begin{align*}
X &= Y = 0; \\
X &:= 1; \quad Y := 1; \\
a &:= Y; \quad b := X;
\end{align*}
\]