C/C++ Concurrency and Program Transformations
Recap: Transformations

Redundant access elimination such as

- \( Y_{rlx} = v; Y_{rlx} = v' \leadsto Y_{rlx} = v' \) (OW)
- \( Y_{rlx} = v; a = Y_{rlx} \leadsto Y_{rlx} = v; a = v; \) (RAW)
- \( a = Y_{rlx}; a = Y_{rlx} \leadsto a = Y_{rlx} \) (RAR)

Fence Insertion such as

- \( C_1; C_2 \leadsto C_1; F_{rel,acq,sc}; C_2 \)

Access strengthening such as

- \( Y_{rlx} = v \leadsto Y_{rel} = v \)
- \( a = Y_{rlx} \leadsto a = Y_{acq} \)

Reordering of accesses such as

- \( a = Y_{rlx}; Z_{rlx} = v \leadsto Z_{rlx} = v; a = Y_{rlx} \)
Recap: Correct Compilation

For each consistent execution of the target program there exists a corresponding consistent execution of the source program with same outcome.

\[ X \in \llbracket P_{tgt} \rrbracket. \exists X' \in \llbracket P_{src} \rrbracket. O(X) = O(X') \]
Recap: Proof Strategy

1. For each consistent execution $X'$ of target program $P_{tgt}$, define an execution $X$ of $P_{src}$.
2. Show that $X$ is consistent.
3. Show that $O(X) = O(X')$.
Recap: Overwritten Write Elimination

We proved the correctness of overwritten write transformation in C11.

Overwritten write:

\[ Y_o = v; Y_o' = v' \sim Y_o' = v' \]  (OW)

where \( o, o' \in \{ \text{na}, \text{rlx}, \text{rel}, \text{sc} \} \)

**Theorem**

(OW) is correct when \( o \sqsubseteq o' \).
Correctness of Read-after-Write Elimination

\[ Y_\circ = v; a = Y_\circ' \sim \Rightarrow Y_\circ = v; a = v; \]  

**Theorem**

*(RAW)* is correct when \( o' \sqsubseteq o \)

Remember the proof steps:

1. For each consistent execution \( X' \) of target program \( \mathbb{P}_{tgt} \), define an execution \( X \) of \( \mathbb{P}_{src} \).
2. Show that \( X \) is consistent.
3. Show that \( O(X) = O(X') \)
Correctness of Read-after-Write Elimination

Let $X'$ be a target execution where $Y_o = v$ results in event $w = W_o(Y, v)$.
We define source execution $X$ where $a = Y_o'$ results in event $r = R_o'(Y, v)$ such that

- $X.E = X'.E \cup \{r\}$
- $X.po = X'.po \cup \{(w, r)\} \cup \{(a, r) \mid X'.po(a, w)\} \cup \{(r, b) \mid X'.po(w, b)\}$
- $X.rf = X'.rf \cup \{(w, r)\}$
- $X.mo = X'.mo$
- $X.sc = X'.sc \cup \{(w, r) \mid o' = sc\} \cup \{(p, r) \mid X'.(p, w) \land o' = sc\} \cup \{(r, q) \mid X'.sc(w, q) \land o' = sc\}$
Correctness of Read-after-Write Elimination

- $X.E = X'.E \cup \{r\}$
- $X.po = X'.po \cup \{(w, r)\} \cup \{(a, r) \mid X'.po(a, w)\} \cup \{(r, b) \mid X'.po(w, b)\}$
- $X.rf = X'.rf \cup \{(w, r)\}$
- $X.mo = X'.mo$
- $X.sc = X'.sc \cup \{(w, r) \mid o' = sc\} \cup \{(p, r) \mid X'.(p, w) \land o = sc\} \cup \{(r, q) \mid X'.sc(w, q) \land o = sc\}$
Now we prove the theorem similar to OW theorem. 
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X.\text{hb} = X'.\text{hb} \cup \{(w, r)\} \cup \{(a, r) \mid X'.\text{hb}(a, w)\} \cup \{(r, b) \mid X'.\text{hb}(w, b)\}$$
RAW Elimination: Proof of Correctness

Now we prove the theorem similar to OW theorem. Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X.hb = X'.hb \cup \{(w, r)\} \cup \{(a, r) \mid X'.hb(a, w)\} \cup \{(r, b) \mid X'.hb(w, b)\}$$

(irrHB)

Assume there exists a $X.hb$ cycle.

It implies that there exists a $(X.po \cup X.sw)^+$ cycle.

It implies that the $(X.po \cup X.sw)^+$ cycle passes through event $r$. However, from definition, $(w, r) \in X.sw; [r] \subseteq X.po$ and there is no outgoing $\{r\}; X.sw$ edge.

Therefore $X.hb$ through $r$ contains $X.po(-, r)$ and $X.po(r, -)$.

It implies $X'.hb$ is a cycle which is a contradiction.

Therefore $X.hb$ satisfies (irrHB).
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$X.hb = X'.hb \cup \{(w, r)\} \cup \{(a, r) \mid X'.hb(a, w)\} \cup \{(r, b) \mid X'.hb(w, b)\}$

Assume $X$ is not coherent.
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$X.hb = X'.hb \cup \{(w, r)\} \cup \{(a, r) \mid X'.hb(a, w)\} \cup \{(r, b) \mid X'.hb(w, b)\}$

Assume $X$ is not coherent.
Assume there exists a $X.rf; X.hb$ cycle.
It involves event $r$.
It implies $X.rf; \{r\}; X.po$ cycle.
However, from definition, this is a contradiction in $X$.
Therefore $X$ satisfies (CohHBXF).
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X.\text{hb} = X'.\text{hb} \cup \{(w, r)\} \cup \{(a, r) | X'.\text{hb}(a, w)\} \cup \{(r, a) | X'.\text{hb}(w, a)\}$$

Assume $X$ is not coherent.
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X\.hb = X'\.hb \cup \{(w, r)\} \cup \{(a, r) \mid X'\.hb(a, w)\} \cup \{(r, a) \mid X'\.hb(w, a)\}$$

Assume $X$ is not coherent.
Assume there exists a $X\.mo; X\.rf; X\.hb$ cycle.
It involves event $r$.
Considering the outgoing edges from $r$:

$$X\.po(r, b) \implies X\.po(w, b).$$

Therefore a $X\.mo; X\.rf; X\.hb$ cycle implies that there already exists a $X'\.mo; X'\.rf; X'\.hb$ cycle which is a contradiction.

Therefore $X$ satisfies (Coh-RW).
Remember the proof strategy steps:

1. Show that \( X \) is consistent.

We show \( X \) is consistent by contradiction.

Note that
\[
X \cdot \text{hb} = X' \cdot \text{hb} \cup \{(w, r)\} \cup \{(a, r) \mid X' \cdot \text{hb}(a, w)\} \cup \{(r, b) \mid X' \cdot \text{hb}(w, b)\}
\]

Assume \( X \) is not coherent.

Exercise: Prove rest of the coherence axioms.
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

(Atomicity)
Assume $X$ does not preserve (Atomicity).
However, from definition, it implies $X'$ violates (Atomicity) which is a contradiction.
Therefore $X$ preserves atomicity.
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

(ConsSC)

Two possibilities:

Case $X.sc = X'.sc$ when $'o \neq sc$

In this case $X$ preserves (ConsSC) as $X'$ preserves (ConsSC).

Case $X'.sc \subseteq X.sc$ when $'o' = sc$

Assume $X$ does not preserve (ConsSC) due to $r$.

However, from definition, it implies $X'$ violates (ConsSC) which is a contradiction. (Why?)

Therefore $X$ preserves (ConsSC).
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

(SCread)

Assume $X$ does not preserve (SCread).

Two possibilities:

**Case** $o' \neq \text{sc}$

From the definition $X$ preserves (SCread).

**Case** $o' = \text{sc}$

In this case $\text{sc}_{\text{imm}}(w, r)$ holds and therefore (SCread) holds.

Therefore source execution $X$ is consistent.
Remember the proof strategy steps:

1. Show that $O(X) = O(X')$

From definition, $X.mo = X'.mo$ and hence $O(X) = O(X')$.

Therefore the transformation is correct.
Correctness of Read-after-Read Elimination

\[ a = Y_{o'}; b = Y_o \leadsto a = Y_{o'}; b = a; \]  

Theorem

(RAR) is correct when \( o \subseteq o' \)

Remember the proof steps:

1. For each consistent execution \( X' \) of target program \( P_{tgt} \), define an execution \( X \) of \( P_{src} \).
2. Show that \( X \) is consistent.
3. Show that \( O(X) = O(X') \)
Correctness of Read-after-Read Elimination

Let $X'$ be a target execution where $a = Y_o'$ results in event $e = R_o'(Y, \nu)$ reading from the write event $w$.
We define source execution $X$ where $b = Y_o$ results in event $r = R_o(Y, \nu)$ such that

- $X.E = X'.E \cup \{r\}$
- $X.po = X'.po \cup \{(e, r)\} \cup \{(a, r) \mid X'.po(a, e)\} \cup \{(r, b) \mid X'.po(e, b)\}$
- $X.rf = X'.rf \cup \{(w, r) \mid X'.rf(w, e)\}$
- $X.mo = X'.mo$
- $X.sc = X'.sc \cup \{(e, r) \mid o = sc\} \cup \{(p, r) \mid X'(p, e) \land o = sc\} \cup \{(r, q) \mid X'.sc(e, q) \land o = sc\}$
Correctness of Read-after-Read Elimination

- $X.E = X'.E \cup \{r\}$
- $X.po = X'.po \cup \{(e, r)\} \cup \{(a, r) \mid X'.po(a, e)\} \cup \{(r, b) \mid X'.po(e, b)\}$
- $X.rf = X'.rf \cup \{(w, r) \mid X'.rf(w, e)\}$
- $X.mo = X'.mo$
- $X.sc = X'.sc \cup \{(e, r) \mid o = sc\} \cup \{(p, r) \mid X'.(p, e) \land o = sc\} \cup \{(r, q) \mid X'.sc(e, q) \land o = sc\}$

![Diagram of X'](target)

![Diagram of X](source)
Now we prove the theorem similar to OW and RAR theorems. Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X.hb = X'.hb \cup \{(e, r)\} \cup \{(a, r) \mid X'.hb(a, e)\} \cup \{(r, b) \mid X'.hb(e, b)\}$$
RAR Elimination: Proof of Correctness

Now we prove the theorem similar to OW and RAR theorems. Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X\cdot hb = X'\cdot hb \cup \{(e, r)\} \cup \{(a, r) \mid X'\cdot hb(a, e)\} \cup \{(r, b) \mid X'\cdot hb(e, b)\}$$

(irrHB)

Assume there exists a $X\cdot hb$ cycle.

It implies that there exists a $(X\cdot po \cup X\cdot sw)^+$ cycle.

It implies that the $(X\cdot po \cup X\cdot sw)^+$ cycle passes through event $r$.

However, from definition, $X\cdot sw(a, r) \implies X\cdot sw(a, e)$ for any event $a$.

Therefore $X\cdot hb$ cycle implies $X'\cdot hb$ is a cycle which is a contradiction.

Therefore $X\cdot hb$ satisfies (irrHB).
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

\[ X \cdot \text{hb} = X' \cdot \text{hb} \cup \{(e, r)\} \cup \{(a, r) \mid X' \cdot \text{hb}(a, e)\} \cup \{(r, b) \mid X' \cdot \text{hb}(e, b)\} \]

Assume $X$ is not coherent.
RAR Elimination: Proof of Correctness

Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that
$$X\cdot\text{hb} = X'\cdot\text{hb} \cup \{(e, r)\} \cup \{(a, r) \mid X'\cdot\text{hb}(a, e)\} \cup \{(r, b) \mid X'\cdot\text{hb}(e, b)\}$$

Assume $X$ is not coherent.
Assume there exists a $X\cdot\text{rf}; X\cdot\text{hb}$ cycle.
It involves event $r$.
It implies $X\cdot\text{rf};\{r\}; X\cdot\text{po}$ cycle.
However, from definition, in this case there is a this is a $X'\cdot\text{rf}; X'\cdot\text{hb}$ cycle through $e$ which is a contradiction.
Therefore $X$ satisfies (CohHB-RF).
RAR Elimination: Proof of Correctness

Remember the proof strategy steps:

1. Show that \( X \) is consistent.

We show \( X \) is consistent by contradiction.

Note that
\[
X.\text{hb} = X'.\text{hb} \cup \{(e, r)\} \cup \{(a, r) \mid X'.\text{hb}(a, e)\} \cup \{(r, b) \mid X'.\text{hb}(e, b)\}
\]

Assume \( X \) is not coherent.
RAR Elimination: Proof of Correctness

Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$X.hb = X'.hb \cup \{(e, r)\} \cup \{(a, r) | X'.hb(a, e)\} \cup \{(r, b) | X'.hb(e, b)\}$

Assume $X$ is not coherent.
Assume there exists a $X.mo; X.rf; X hb$ cycle.
It involves event $r$.
Considering the outgoing edges from $r$:
$X.po(r, b) \implies X.po(e, b)$.
Therefore a $X.mo; X.rf; X hb$ cycle implies that there already exists a a $X'.mo; X'.rf; X'.hb$ cycle which is a contradiction.
Therefore $X$ satisfies (Coh-RW).
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

Note that

$$X.\text{hb} = X'.\text{hb} \cup \{(e, r)\} \cup \{(a, r) \mid X'.\text{hb}(a, e)\} \cup \{(r, b) \mid X'.\text{hb}(e, b)\}$$

Assume $X$ is not coherent.

Exercise: Prove rest of the coherence axioms.
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

(Atomicity)
Assume $X$ does not preserve (Atomicity).
However, from definition, it implies $X'$ violates (Atomicity) which is a contradiction.
Therefore $X$ preserves atomicity.
RAR Elimination: Proof of Correctness

Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

$(\text{ConsSC})$

Two possibilities:

Case $X.sc = X'.sc$ when $o \neq sc$

In this case $X$ preserves $(\text{ConsSC})$ as $X'$ preserves $(\text{ConsSC})$.

Case $X'.sc \subseteq X.sc$ when $o = sc$

Assume $X$ does not preserve $(\text{ConsSC})$ due to $r$.

However, from definition, it implies $X'$ violates $(\text{ConsSC})$ which is a contradiction. (Why?)

Therefore $X$ preserves $(\text{ConsSC})$. 
Remember the proof strategy steps:

1. Show that $X$ is consistent.

We show $X$ is consistent by contradiction.

(SCread)
Assume $X$ does not preserve (SCread).
Two possibilities:

**Case** $o \neq sc$
From the definition $X$ preserves (SCread).

**Case** $o = sc$
In this case $sc_{imm}(e, r)$ holds and $e.ord = sc$ and therefore (SCread) holds.

Therefore source execution $X$ is consistent.
Remember the proof strategy steps:

1. Show that $O(X) = O(X')$

From definition, $X.mo = X'.mo$ and hence $O(X) = O(X')$.

Therefore the transformation is correct.
Show that the following transformation is safe.

\[ a = Y_{\text{na}}; \ b = Y_{o} \sim a = Y_{\text{na}}; \ b = a; \]  

(RAR)